On heat kernel methods and curvature asymptotics for certain cohomogeneity one Riemannian manifolds

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Abstract

We study problems related to the metric of a Riemannian manifold with a particular focus on certain cohomogeneity one metrics. In Chapter 2 we study a set of cohomogeneity one Einstein metrics found by A. Dancer and M. Wang. We express these in terms of elementary functions and find explicit sectional curvature formulae which are then used to determine sectional curvature asymptotics of the metrics. In Chapter 3 we construct a non-standard parametrix for the heat kernel on a product manifold with multiply warped Riemannian metric. The special feature of this parametrix is that it separates the contribution of the warping functions and the heat data on the factors; this cannot be achieved via the standard approach. In Chapter 4 we determine explicit formulae for the resolvent symbols associated with the Laplace Beltrami operator over a closed Riemannian manifold and apply these to motivate an alternative method for computing heat trace coefficients. This method is entirely based on local computations and to illustrate this we recover geometric formulae for the heat coefficients. Furthermore one can derive topological identities via this approach; to demonstrate this application we find explicit formulae for the resolvent symbols associated with Laplace operators on a Riemann surface and recover the Riemann-Roch formula. In the final chapter we report on an area of current research: we introduce a class of symbols for pseudodifferential operators on simple warped products which is closed under composition. We then extend the canonical trace to this setting, using a cut - off integral, and find an explicit formula for the extension in terms of integrals over the factor.

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Chapter 1

Introduction

In this thesis we consider a variety of problems in Riemannian geometry, both on non - compact as well as compact spaces. The aim in most cases is to obtain more explicit formulae in order to conduct computations; frequently we shall assume special symmetry to attain this goal. The main body of the thesis consists of four chapters. We start off in Chapter 2 by conducting an asymptotic analysis for the sectional curvature of a set of cohomogeneity one Einstein metrics constructed in [11] by A. Dancer and M. Wang via the Hamiltonian formalism. In Chapter 3 we construct a non-standard parametrix for the heat kernel on a Riemannian product manifold with multiply warped metric, with the aim of isolating the effect of the warping functions on the heat kernel. Chapter 4 is about resolvent symbols of Laplace operators and their applicability, in particular to heat trace computations. We find explicit formulae for these in the context of a closed Riemannian manifold of arbitrary dimension and apply the result to compute heat trace coefficients; though computationally involved this approach has the advantage that it uses only local data. Resolvent symbols can also be used to derive index formulae; to demonstrate this we determine explicit formulae for Laplace operators over a Riemann surface and recover the Riemann Roch formula. Again this method is local and, moreover, does not rely on the Getzler rescaling. Finally Chapter 5 turns the focus on a class of symbols for pseudodifferential operators on simple warped products; we define symbols of what we call *log - polyhomogeneous radial growth* and show that the class is closed under symbol composition. We then extend the canonical trace to this setting, using a cut - off integral, and study its basic properties. We also investigate in some detail the symbol expansion of an example which is of particular interest to us, namely the resolvent and complex powers of the Laplace - Beltrami operator on a warped product.

Let us now elaborate a little on each topic, a more detailed introduction is given at the start of the individual chapters.

Chapter 2: Sectional curvature asymptotics for certain non-compact cohomogeneity one Einstein metrics. One says that a Riemannian manifold (M, g) is an Einstein manifold if its Ricci tensor Ric is proportional to the metric, that is

$$\mathrm{Ric} = \lambda g$$

for some constant λ. The origin of this condition is to be found in Einstein's field equations describing general relativity, however the study of this structure is interesting from the purely mathematical viewpoint as well. For instance, Einstein metrics on compact manifolds provide critical points of the scalar curvature functional. First examples of Einstein manifolds are Euclidean space $\mathbb{R}^n$ which is Ricci - flat and hence an Einstein manifold with $\lambda = 0$, the unit sphere S^n with the round metric is a compact Einstein manifold with $\lambda = n - 1 > 0$, whilst hyperbolic space $\mathbb{H}^n$ with the canonical metric provides an example of a non-compact Einstein manifold with $\lambda < 0$. All these spaces are homogeneous in the sense that one can identify for each case a Lie group that acts transitively by isometries. One step higher up in complexity are cohomogeneity one manifolds where a compact Lie group acts by isometries such that the principal orbits have codimension one. This simplification is mathematically appealing as it reduces the Einstein equations to a non-linear system of ODEs in the coordinate transverse to the orbits. If we denote this coordinate by t then the metric takes the form

$$\epsilon dt^2 + g_t$$

where g_t denotes a metric on the principal orbit that varies in the parameter t and $\epsilon = 1$ in the case Riemannian manifolds whilst $\epsilon = -1$ in case the underlying manifold is Lorentzian. From the physical point of view the cohomogeneity one assumption provides a fruitful testing ground since, away from special orbits, a cohomogeneity one Einstein manifold yields a spatially homogeneous Lorentz Einstein manifold, in fact the Schwarzschild metric and the Taub-NUT metric satisfy the cohomogeneity one condition. The first case of a Riemannian (as opposed to Lorentzian) cohomogeneity one Einstein manifold that is not homogeneous was constructed in [35] by D. Page on the non-trivial S^2 - bundle over S^2 with respect to a $U(2)$ - action, with principal orbit $S^3 \cong U(2)/U(1)$; a result that motivated L. Bérard Bergery to study the underlying mathematical structure of cohomogeneity one Einstein manifolds in its own right and to find new examples, both of compact as well as complete non - compact type [3].

In [9] and [11] A. Dancer and M. Wang investigate the relationship between notions of integrability in Hamiltonian systems and solutions to the cohomogeneity one Einstein equations. They find that under certain assumptions (such as the presence of a strictly lower - dimensional special orbit or the isotropy representation of the principal orbit decomposing into distinct subrepresentations) the latter equations are equivalent to the Hamiltonian flow on the zero - energy surface of a Hamiltonian H whose kinetic energy term is an indefinite non - degenerate quadratic form. Furthermore, in particular cases they find non-trivial functions F, ϕ that satisfy the equation $\{F, \mathsf{H}\} = \phi \mathsf{H}$ (where $\{,\}$ denotes the Poisson bracket). As the phase space in these cases is of low dimension this makes the system integrable on the zero set of the Hamiltonian. In this way they find new cohomogeneity one Einstein metrics.

In this chapter we study the sectional curvature of some of these metrics as

$t \to \infty$. First we find a new representation for the solutions given in [11] using only elementary functions. Then, to calculate the sectional curvature we make use of the fact that cohomogeneity one metrics are Riemannian submersions, thus the formulae of O'Neill apply. In the particular case at hand we are dealing with doubly warped products; such metrics take the form $dt^2 + f_1^2(t)g_1 + f_2^2(t)g_2$ with f_1, f_2 smooth functions (called the warping functions or warping factors) depending on the transverse variable only and g_1, g_2 fixed background metrics. This enables us to find explicit and simple formulae for the sectional curvature in terms of the warping functions which are then applied to the particular case at hand to directly compute the long - time sectional curvature limit.

Chapter 3: A non-standard parametrix for the heat kernel on multiply warped products. Here we generalise the construction of a parametrix for the heat kernel on multiply warped products, first proposed by P.C. Lue [29] in the context of simple warped products. The parametrix studied here differs from the standard approach, let us briefly describe the latter: we recall that the heat kernel is a fundamental solution to the heat equation on a Riemannian manifold (M, g); that is a continuous function $s(t, x, y)$ on $(0, \infty) \times M \times M$ that is continuously differentiable in t, twice continuously differentiable in x, y, satisfies the heat equation

$$(\partial_t + \triangle_y)s = 0 \tag{1.0.1}$$

(here $\triangle_y$ denotes the Laplace - Beltrami operator with respect to the variable y) and finally has the property that $\lim_{t\to 0} s(t, x, \cdot) = \delta_x$ (the Dirac - delta distribution based at $x \in M$). In some cases this function can be written down explicitly, such as for Euclidean space $\mathbb{R}^n$ where

$$s(t, x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\{-\frac{\|x - y\|^2}{4t}\}\,. \tag{1.0.2}$$

However generically one needs to revert to indirect methods to study this object, and one way to do so is by constructing an approximation to the heat kernel (also referred to as a parametrix). Such a parametrix is sufficient to study the short time behaviour of the trace of the heat kernel which provides geometric information about the underlying manifold. Formally it is a smooth function $p(t,x,y)$ on $(0,\infty)\times M\times M$ such that $(\partial_t+\triangle_y)p$ extends to a continuous function on $[0,\infty)\times M\times M$ and such that $\lim_{t\to 0} p(t,x,\cdot)=\delta_x$ is the Dirac - delta distribution based at $x\in M$ (it is helpful to compare these conditions for $p(t,x,y)$ with the defining properties of the heat kernel $s(t,x,y)$ to see the similarities of the two objects). The standard approach to its construction was introduced by S. Minakshisundaram and A. Pleijel for compact Riemannian manifolds without boundary in [31]: based on the premise that the heat kernel on a Riemannian manifold ought to be a perturbation of the Euclidean heat kernel (1.0.2) (at least for a small initial time period) the idea is to start with an expression of the form

$$H_k(t,x,y) := \underbrace{\frac{1}{(4\pi t)^{n/2}}\exp\{-\frac{\rho^2(x,y)}{4t}\}}_{\text{Euclidean form of the heat kernel}}\cdot\big(U_0(x,y)+U_1(x,y)t+\cdots U_j(x,y)t^k\big) \tag{1.0.3}$$

where ρ denotes the Riemannian distance. These functions should approximate the heat kernel, in particular they should "almost" solve the heat equation. Formally this is implemented by demanding that

$$(\partial_t+\triangle_y)H_k(\cdot,x,\cdot) = \frac{1}{(4\pi t)^{n/2}}\exp\{-\frac{\rho_x^2(y)}{4t}\}\,\triangle_y\,U_k(x,y)t^k\,,$$

i.e. all the terms vanish when the heat operator is applied except for the t^k - coefficient (i.e. the highest power in t). This condition gives rise to a recursive system of differential equations in the coefficient functions U_j which can then be solved (note from the right hand side of (1.0.3) inherently yields an expansion in powers of t). Having constructed the series $H_\infty(t,x,y)$ one can deduce a short time expansion of the heat trace:

$$\int_M s(t,x,x)\,d\mu(x) \backsim_{t\to 0} \sum_{j\geq 0}\frac{1}{(4\pi)^{\frac{n}{2}}}\int_M U_j(x)\,d\mu(x)\;t^{-\frac{n}{2}+j} \tag{1.0.4}$$

and the geometric insight here is contained in the coefficients U_j, for example $\int_M U_0(x)d\mu(x)$ is equal to the Riemannian volume of M, and in the case where M is a surface we have $\int_M U_1(x)d\mu(x) = \pi\chi(M)/3$ with $\chi(M)$ the Euler characteristic of M.

Now this method works well for compact manifolds and a generic metric g. Let us then consider a product manifold

$$M = I \times \mathcal{M}_1 \times \mathcal{M}_2 \quad \text{with metric} \quad dr^2 + f_1^2(r)g_1 + f_2^2(r)g_2 \tag{1.0.5}$$

where $(\mathcal{M}_i, g_i)$ for $i = 1, 2$ are compact Riemannian manifolds, I is an open interval and the warping functions $f_1, f_2 \colon I \to (0, \infty)$ are smooth positive functions. Naturally one would like to know whether an expansion similar to (1.0.4), parametrised in r, can be obtained and to what extend its coefficient functions $U_j\big((r, x), (r', y)\big)$ can be factored into terms of the warping functions f_1, f_2 and the coefficients from the expansion (3.1.6) on $\mathcal{M}_1 \times \mathcal{M}_2$. It turns out that the answer to the second part of the question, if one uses the standard parametrix construction described above, is"very limited" - already in the setting of simple warps. This was pointed out by Ping-Charng Lue in [29] and motivated him to study an alternative construction which provides a parametrix where the contributions of the warping function f and the contribution from the fibre $\mathcal{M}$ are more explicit. In this chapter we generalise this approach to multiply warped products. The main result, shown in Section 3.3, is that the resulting parametrix as well as the structural features of the proof in [29] adapt to this case and that the newly arising features, compared to single warps are due to the fact that the coefficients U_j are now polynomials in several eigenvalues coming from distinct factors, requiring additional care so as to maintain the essential estimates in the proofs.

Chapter 4: Explicit formulae for resolvent symbols and their application

In this chapter we step away from non-compact manifolds to compact manifolds without boundary and motivate a new approach for deriving heat trace coefficients directly. The method is based on resolvent symbols, in particular it avoids the use of global estimates on the heat kernel, the latter approach was taken by H.P. McKean and I.M. Singer in [24]. Resolvent symbols also facilitate the derivation of index formulae without the use of Getzler's rescaling (i.e. the reduction of the proof to the case of a generalised harmonic oscillator via a rescaling of both the space and Clifford variables) and we shall demonstrate both applications in this chapter.

Concretely, let (M, g) be a compact Riemannian manifold without boundary of dimension n and denote by $\triangle$ the corresponding Laplace Beltrami operator. The heat kernel was described for the purposes of the previous chapter as the fundamental solution $s(t, x, y)$ to the heat equation $\partial_t + \triangle_y$ on M; here we shall use the equivalent formulation as the Schwartz kernel $k_\triangle(t, x, y)$, of the heat operator $e^{-t\triangle}$. The latter is defined as a Cauchy integral via the holomorphic functional calculus by

$$e^{-t\triangle} := \frac{i}{2\pi} \int_\gamma e^{-t\lambda} (\triangle - \lambda)^{-1} \, d\lambda \tag{1.0.6}$$

where the contour γ properly encloses the positive real axis which contains the spectrum of $\triangle$. The connection between the operator and its Schwartz kernel is that the latter is, in general, a family of distributions over M, parametrised in x and t, satisfying the equation

$$(e^{-t\triangle} f)(x) = \langle k_\triangle(t, x, \cdot), f \rangle \qquad \big(f \in C^\infty(M)\big)$$

(though for the heat operator the Schwartz kernel identifies with a smooth function). Locally $k_\triangle$ is given by an oscillatory integral

$$k_\triangle(t, x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \sigma(x, \xi) \, d\xi$$

with local symbol

$$\sigma(x, \xi) = \frac{i}{2\pi} \int_\gamma e^{-t\lambda} r(x, \xi, \lambda) d\lambda$$

where $r(x, \xi, \lambda)$ in turn denotes the local symbol of the resolvent operator. The latter admits an asymptotic expansion

$$r(x, \xi, \lambda) \backsim \sum_{j \geq 0} r_{-2-j}(x, \xi, \lambda)$$

which is valid for $|\xi| + |\lambda|^{1/2} \geq 1$ and λ in a suitable sector $\Lambda \subset \mathbb{C}$. This expansion is of central importance to the chapter, the terms on the right hand side are the resolvent symbols mentioned above and we shall study these in detail. In particular we provide explicit formulae for the first terms in the asymptotic expansion, to the best of our knowledge these do not appear elsewhere in the literature. These closed formulas facilitate a direct and elementary calculation of the heat coefficients in the short-time asymptotic expansion of the heat trace

$$\operatorname{Tr}\left(e^{-t\triangle}\right) = \int_M \operatorname{tr}\left(k_\triangle(t,x,x)\right)\,dx \quad \backsim_{t\to 0_+} \quad \sum_{j\geq 0} c_j\, t^{\frac{j-n}{2}} \tag{1.0.7}$$

via well - known formulas for the coefficients c_j (here tr refers to the usual trace defined on matrices and dx locally identifies with Lebesgue measure). The coefficients with odd index are known to vanish whilst those with an even index are given by

$$c_{2k} = \int_M \operatorname{tr}\left(c_{2k}(x)\right)\,dx \tag{1.0.8}$$

where

$$c_{2k}(x) = \int_{\mathbb{R}^n}\int_\gamma e^{-\lambda}\, r_{-2-2k}(x,\xi,\lambda)\, đ\lambda đ\xi \tag{1.0.9}$$

(here $đ\lambda = id\lambda/2\pi$ and $đ\xi = d\xi/(2\pi)^n$). We shall illustrate this by applying our result to determine the first three of these integrals and thereby recover well - known geometric identities for these coefficients.

A further application of resolvent symbols is that they are effective for deriving index formulae. As a demonstration of this we determine explicit formulae for the resolvent symbols of Laplace operators defined over a Riemann surface and recover the Riemann-Roch formula, again by a direct and elementary calculation. Let us briefly outline the approach: let M be a smooth compact manifold without boundary of even dimension $n = 2k$ with vector bundles $\mathcal{E}^\pm$ over M and consider a first - order elliptic differential operator

$$D\colon C^\infty(M,\mathcal{E}^+) \to C^\infty(M,\mathcal{E}^-) \tag{1.0.10}$$

acting on smooth sections. It was observed by H.P. McKean and I.M. Singer in [24] that the index of D (i.e. the integer given by difference of the dimension of the kernel and cokernel of D, denoted $\operatorname{ind} D$) identifies with the heat trace of the Laplacians $\triangle = D^*D$ and $\widetilde{\triangle} = DD^*$:

$$\operatorname{ind} D = \operatorname{Tr}(e^{-t\triangle}) - \operatorname{Tr}(e^{-t\widetilde{\triangle}}) = \int_M \operatorname{tr}\Big(k_\triangle(t,x,x)\Big) - \operatorname{tr}\Big(k_{\widetilde{\triangle}}(t,x,x)\Big)\ |dx|\ . \tag{1.0.11}$$

On the other hand, if M is a Riemannian spin manifold and D is of Dirac - type, that is

$$D = \not{D} \otimes I + I \otimes \nabla^{\mathcal{F}} : \quad C^\infty(M, \mathcal{S}^+ \otimes \mathcal{F}) \longrightarrow C^\infty(M, \mathcal{S}^- \otimes \mathcal{F})$$

where $\mathcal{S}^\pm$ denotes the spinor bundle and $\mathcal{F} \to M$ is some coefficient bundle, then the Atiyah-Singer index theorem states that

$$\operatorname{ind} D = \frac{1}{(2\pi)^{n/2}} \int_M \widehat{A}(M)\operatorname{ch}(\mathcal{F}) \tag{1.0.12}$$

where $\widehat{A}(M)$ is the $\widehat{A}$ - genus form with respect to Riemannian curvature R whilst $\operatorname{ch}(\mathcal{F})$ denotes the Chern character of the coefficient bundle $\mathcal{F}$. There are different approaches to proving the identity (1.0.12). One is to use the McKean - Singer formula (1.0.11) and the short time asymptotic expansion

$$\operatorname{tr}(k_\triangle(t,x,x)) \ \backsim_{t\to 0_+} \ \sum_{j\geq 0} c_j(x) t^{\frac{j-n}{2}} \tag{1.0.13}$$

of the heat kernel along the diagonal as follows. One first substitutes (1.0.13) into the right hand side of (1.0.11) to get

$$\operatorname{ind} D = \sum_{j\geq 0} \int_M c_j(x) - \widetilde{c}_j(x) dx\, t^{\frac{j-n}{2}} \tag{1.0.14}$$

for t small (here $\widetilde{c}_j(x)$ refers to the coefficients of the asymptotic expansion of $\operatorname{Tr}(e^{-t\widetilde{\triangle}})$.) Since the left hand side does not depend on t one can take $t \to 0_+$ and obtain a finite expression. The task is then to identify the expression

$$c_0(x) - \widetilde{c}_0(x)\, dx$$

coming to the constant coefficient in the expansion with the local index density on the right hand side of (1.0.12).

The proof by McKean and Singer in [24] in conjunction with [31] uses global estimates of the heat kernel (see also [22] for the case of the Riemann Roch theorem). Alternatively, a rescaling of variables can be applied, leading to a transformation of the Laplacian operator into a generalized harmonic oscillator for which an identification of the corresponding heat kernel with the index density is known (this method is referred to as Getzler rescaling as it was introduced by E. Getzler in [13]). Instead we study an alternative, more elementary method. The key input are the formulae (1.0.8)-(4.1.5) for the heat coefficients and an explicit knowledge of the resolvent symbols r_{-2-2k} which appear there. Substituting these into (1.0.14) and take $t \to 0_+$ yields

$$\operatorname{ind} D = \int_M \left(\int_{\mathbb{R}^n} \int_\gamma e^{-\lambda} \{\operatorname{tr} r_{-2-2j}(x,\xi,\lambda) - \operatorname{tr} \widetilde{r}_{-2-2j}(x,\xi,\lambda)\}\, đ\lambda đ\xi \right) dx\,. \quad (1.0.15)$$

The task is then to derive the equality of densities by relating the integrand above to the topological index density. Compared to existing methods, this approach has the advantage that it computes the index directly from the first n terms of the local symbols of the resolvent operator. These are polynomials whose coefficients are determined by the local symbol of the Laplacians, together with a finite number of its derivatives. Thus it reflects the local nature of the index quite well.

In Section 4.3 we shall study this technique using as a concrete example the Riemann-Roch-Hirzebruch theorem. Section 4.3.1 sets out the context of the theorem, then in Section 4.3.2 we determine explicit formulae for the resolvent symbols of Laplace operators defined over a Riemann surface. These are then applied to derive the Riemann-Roch formula in Section 4.3.3, again by a direct and elementary calculation. As in the case of heat trace coefficients for the Laplace Beltrami operator in the first part of this chapter, the explicit form of our formulae and the method to derive the Riemann Roch theorem are new in the literature.

Chapter 5: On log-polyhomogeneous symbols over simple warped products. In this chapter we establish an extension of the canonical trace on log - polyhomogeneous pseudodifferential operators as considered by M. Lesch in [27] to a suitable class of pseudodifferential operators over single warped Riemannian product manifolds.

Let $\pi\colon E \to M$ be a smooth vector bundle over an n-dimensional closed Riemannian manifold M and consider a classical pseudodifferential operator (ψdo) $A\colon C^\infty(M;E) \to C^\infty(M;E)$ with local symbol σ. If A has non-integer order then the canonical trace $\mathrm{TR}(A)$ is defined by the formula

$$\mathrm{TR}(A) := \int_M \mathrm{TR}_x(A)\,dx \tag{1.0.16}$$

where dx locally identifies with Lebesgue measure and

$$\mathrm{TR}_x(A) := \,-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}_x\left(\sigma(x,\xi)\right) đ\xi \tag{1.0.17}$$

is a finite - part integral (the finite part integral systematically ignores divergent terms in the following way: one shows that there exists an asymptotic expansion of the integral $\int_{B_x^*(R)} \mathrm{tr}_x\left(\sigma(x,\xi)\right) đ\xi$ where $B_x^*(R)$ denotes the ball in T_x^*M centered at the origin of radius R (the expansion is in terms of R); and then the finite part integral is defined to be the constant coefficient in this expansion. In this way one discards the divergent terms of $\int_{T_x^*M} \mathrm{tr}_x\left(\sigma(x,\xi)\right) đ\xi$. This procedure is similar to the Hadamard regularisation and is sometimes also referred to as a cut - off integral.)

Motivated by the appearance of $\log t$ - powers in the heat trace expansion of the Laplacian in certain contexts, and by the search for a natural algebraic setting of classical pseudodifferential operators with respect to commutator presentations, M. Lesch introduces in [27] a slightly larger class of pseudodifferential operators with log - polyhomogeneous symbols and studies extended notions of the canonical trace (and the residue trace) in this context. The symbols that he considers admit an expansion of the form $a \backsim_{\xi\to\infty} \sum_{j\geq 0} a_{\mu-j}$ where for a fixed k and each j

$$a_{\mu-j}(x,\xi) = \sum_{i=0}^{k} a_{\mu-j,i}(x,\xi)\log^i|\xi|$$

with each $a_{\mu-j,i}$ homogeneous in ξ of degree $\mu - j$. A similar pattern arises when one considers the leading symbol of the resolvent of the Laplace Beltrami operator over a warped product. In this case however, the log - polyhomogeneous expansion arises not only in the ξ - variable but also in the non - compact space variable. This led us to study a generalised class of log - polyhomogeneous symbols: we consider log - polyhomogeneous symbols in the setting of a simple warped product $\mathcal{M} := [0,\infty)\times M$ where (M,g) is a closed Riemannian manifold together with a metric of the form $dr^2 + h^2(r)g$ where $f\colon [0,\infty)\to\mathbb{R}$ is a smooth positive function that diverges to $+\infty$ as $r\to\infty$. For example metric cones are of this form with $f(r) = r^k$ (k a positive integer), such as the polar coordinate representation $dr^2 + r^2 g_{S^{n-1}}$ of the Euclidean metric on $(0,\infty)\times S^{n-1}\cong \mathbb{R}^n\setminus\{0\}$. Another example is hyperbolic space, where the metric takes the form $dr^2 + \sinh^2(r)g_{S^{n-1}}$.

The symbols we study are log - polyhomogeneous in the ξ-variable, as defined by Lesch, and moreover exhibit the log - polyhomogeneous property in the "radial" space variable that parametrises the factor $[0,\infty)$. We show that this class is closed under symbol multiplication and therefore provides a calculus. To study the canonical trace in this setting we consider families of operators over M, parametrised in $r\in[0,\infty)$ and locally defined by our symbols. The canonical trace then arises via a repeated finite - part integral, first with respect to the ξ variable and then with respect to the parameter r. The obstruction to global well - definedness of this parametrised trace on the factor M needs to be taken into account, however there are certain types of operators (analogues to the compact case) in which the obstruction vanishes and for those we identify a formula for the canonical trace in terms of certain finite part integrals over the factor M that resemble the standard canonical trace, except for the presence of an additional dimension in the cotangent space at each point corresponding to the non-compact space variable. With respect to the asymptotic expansion of the leading symbol of the resolvent of the Laplace Beltrami operator mentioned above, this additional dimension defines the hypersurface where the terms in the asymptotic expansion diverge, away from this hypersurface the terms in the expansion decay at least polynomially in r. To illustrate this phenomenon we describe the case of the Laplace Beltrami operator in detail.

Chapter 2

Sectional curvature asymptotics for certain non-compact cohomogeneity one Einstein metrics

2.1 Introduction

An Einstein manifold is a Riemannian manifold (M, g) where the Ricci tensor satisfies the equation $\mathrm{Ric} = \lambda g$ for some constant λ. As mentioned above, the concept originated in Physics from Einstein's theory of general relativity (in the context of Lorentzian manifolds). However, studying Einstein spaces is of interest also from a mathematical point of view. For example on a given compact manifold M, an Einstein metric provides a critical point of the total scalar curvature functional $S[g] = \int_M s_g(x) d\mu_g(x)$ on the space of unit volume metrics. Furthermore, the Einstein condition also provides a good way to distinguish certain metrics as "optimal" similar to the way this is achieved on surfaces by asking for metrics of constant scalar curvature. On Riemannian manifolds of dimension 2 scalar curvature is the only notion of curvature whereas for higher dimensional Riemannian manifolds there is the Riemann curvature tensor (respectively the sectional curvature function), the Ricci curvature tensor and the scalar curvature function. Generalising constant scalar curvature condition to higher dimensions yields a large class of metrics satisfying this constraint; for example on any compact manifold of dimension ≥ 3 the family of Riemannian metrics with constant scalar curvature is infinite - dimensional. On the other hand, if one generalises the constant curvature condition by requiring constant sectional curvature then the resulting class of metrics is very small; in fact for each sign of the constant sectional curvature there is exactly one complete, simply connected Riemannian manifold (up to isometry), namely the sphere S^n with the round metric for sectional curvature $+1$, $\mathbb{R}^n$ for zero sectional curvature and hyperbolic space $\mathbb{H}^n$ with the canonical metric for the case where sectional curvature equals -1. Hence for dimension $n \geq 3$ many manifolds do not admit a metric of constant sectional curvature (more details may be found in [4]). Thus constant scalar curvature is too weak a condition whereas constant sectional curvature is too strong; and one is left with constant Ricci curvature. But Ricci curvature as a function on the unit tangent bundle UM of M is constant precisely when $\mathrm{Ric} = \lambda g$ for some $\lambda \in \mathbb{R}$; this is the Einstein condition.

Now without simplification the Einstein equations $\mathrm{Ric}\, g = \lambda g$ are hard to study so it is natural to start by imposing simplifying assumptions on the metric g, such as possessing large isometry groups. In fact, if one assumes that the metric is homogeneous (i.e. there is an isometric and transitive Lie group action) then the Einstein condition becomes algebraic. Slightly less restrictive is the assumption of cohomogeneity one where the metric g is required to be invariant under the action of a Lie group G that acts properly on the manifold M with principal orbits of codimension one. In this case the Einstein equations reduce to a nonlinear system of ODEs, examples of such metrics were pointed out in Physics by Page [35], which motivated the mathematical generalisation by Bérard-Bergery [3]. But also the Schwarzschild metric, the Eguchi - Hanson metric, the Taub-NUT metric or the cohomogeneity one manifolds of [8] in Physics as well as examples found in Mathematics [10, 42] and more recently [5], just to name a few, illustrate the pervasiveness of cohomogeneity one metrics.

In this chapter we shall study metrics that arise from the consideration of the cohomogeneity one Ricci - flat Einstein equations $\mathrm{Ric}\, g = 0$ as a Hamiltonian system with an additional constraint, an approach taken by Andrew Dancer and McKenzie Wang in [9, 11] to construct new examples of cohomogeneity one Einstein manifolds. The treatment in [11] assumes that the principal orbit is a product $(G_1/K_1) \times (G_2/K_2)$ of distinct isotropy irreducible spaces which means that the metric is diagonal of the form

$$dt^2 + f_1^2(t)\bar{g}_1 + f_2^2(t)\bar{g}_2 \tag{2.1.1}$$

where $\bar{g}_i$ is a homogeneous background metric on the i^{th} component of the principal orbit. For the dimension pairs $(d_1, d_2) = (2, 8), (3, 6)$ and $(5, 5)$, Dancer and Wang find that

$$f_2^{2\frac{d_1+1}{d_1-1}} = K \coth\left(\frac{R}{2}\right) \int \frac{\tanh\left(\frac{R}{2}\right)(\cosh R - 1)^{\frac{d_1+1}{d_1-1}}}{\sinh R}\, dR \tag{2.1.2}$$

and

$$f_1^{(d_1-1)} f_2^2 = \frac{C}{2A_1}(\cosh R - 1) \tag{2.1.3}$$

solve the above system. Here $R = \sqrt{\frac{(d_1-1)A_1}{d_1}} r + \text{const.}$ depends on t via $r' := 1/f_1$ and C, K are non-zero constants. In this chapter we pick up from the representation above and find a description for f_1 and f_2 in terms of elementary functions. These are then used to study sectional curvature asymptotics directly. Let us briefly outline the organisation of the sections: we shall start by recalling the basic notions of cohomogeneity one manifolds in Section 2.2 followed by Section 2.3 where we determine the elementary function representation for f_1 and f_2. We then turn to the study of sectional curvature in section 2.2.2. Since we are dealing with Riemannian submersions, O'Neill's formulae turn out to be particularly helpful to find expressions for the sectional curvature of our cohomogeneity one manifolds in terms of sectional curvature in the principal orbits. Finally, Section 2.4 is concerned with the sectional curvature asymptotics as $t \to \infty$ for the metrics from Section 2.3.

2.2 Cohomogeneity one manifolds

In this section we outline the properties of cohomogeneity one manifolds as introduced by B. Bergery in [3], focusing on those aspects that are important to the study in this thesis.

2.2.1 Definition and basic properties

A connected Riemannian manifold (M, g) is said to be of *cohomogeneity one* (or a cohomogeneity one manifold) with respect to a group G if the latter acts by isometries on M such that the codimension of its principal orbits in M is one. The *systematic* study of cohomogeneity one Riemannian manifolds was initiated by a construction of an Einstein metric on the non-trivial sphere bundle over S^2 by D. Page [35], the group of isometries of that metric is of dimension four and its principal orbit had codimension one. In [3] Berard Bergery generalised this metric by putting the cohomogeneity one property into focus and introduced a theory for such objects in the context of Riemannian geometry in n dimensions. As suggested by this line of development, cohomogeneity one manifolds are interesting for example for the study of Einstein manifolds, i.e. Riemannian manifolds whose metric g satisfies $\mathrm{Ric}(g) = \lambda g$ for some constant λ. Seen as a partial differential equation, the latter condition on the metric is a rather complicated non - linear system, however in the case of a cohomogeneity one manifold it reduces to a system of ordinary differential equations where the independent variable is a coordinate transverse to the orbits. Also, the problem of finding curvature formulae simplifies in the setting of cohomogeneity one Riemannian manifolds. Generally, for a non-homogeneous Riemannian manifold M, (i.e. with non - transitive isometry group) the curvature formulae at a point in a principal orbit (which is a homogeneous space) involve the curvature of the orbit, the curvature of the coset space M/G and cross - terms. However, in the cohomogeneity one case the coset space is one - dimensional and therefore has no curvature, hence the curvature formulae for such spaces simplify.

The coset space M/G is a connected differentiable manifold, and since G acts by isometries it inherits a Riemannian metric as a quotient space relative to which the quotient map $\pi\colon M \to M/G$ is a Riemannian submersion. The principal orbits lie over the interior of M/G whereas the orbits over boundary points (if any) are not principal. Essentially, that is up to isometry, there are only a finite number of forms that M/G can take on: if M is compact then M/G either has no boundary and is isometric to a circle of length ℓ, or it has two boundary points and is isometric to the interval $[0, a]$. On the other hand, if M is not compact then the same is true for M/G (if G is compact). In this case the latter may have no boundary - the possible isometry types for M/G then are the real line $\mathbb{R}$, the ray $(0, \infty)$, or the finite - length interval $(0, a)$. Otherwise M/G has a boundary point and is isometric to the ray $[0, \infty)$ or $[0, a)$. We summarise the possible space forms and associated basic properties below:

$M/G \cong$	**Non-principal orbits**	M **compact**	M **complete**
(S^1, ℓ)	None	Yes	Yes
$[0, a]$	Two	Yes	Yes
$\mathbb{R}$	None	No	Yes
$[0, \infty)$	One	No	Yes
$(0, \infty)$	None	No	No
$(0, a)$	None	No	No
$[0, a)$	One	No	No

In all cases above the model manifolds in the left-most column are understood to carry their respective canonical metric.

Next we briefly discuss a parametrisation for M with reference to M/G and a fixed principal orbit O. Given a point $p \in O$ one can choose a geodesic $\gamma\colon I \to M$ (with $I \subset \mathbb{R}$) passing through p and perpendicular to O. It then is orthogonal to all orbits that it crosses. Indeed, let X be a Killing field with respect to G, this is a vector field whose flow generates an isometry induced by G, in particular the Lie derivative with respect to X of the metric vanishes, that is $\mathcal{L}_X g = 0$. Let D denote the Levi-Civita connection. Starting from the identity

$$\dot\gamma\langle\dot\gamma, X\rangle = \langle D_{\dot\gamma}\dot\gamma, X\rangle + \langle\dot\gamma, D_{\dot\gamma}X\rangle \tag{2.2.1}$$

we know that the first term on the right hand side vanishes since $D_{\dot\gamma}\dot\gamma = 0$. Furthermore, the second term also vanishes since X is Killing. To see this we start with the definition of the Lie derivative

$$\mathcal{L}_X g(Y, Z) = (\mathcal{L}_X g)(Y, Z) + g(\mathcal{L}_X Y, Z) + g(Y, \mathcal{L}_X Z)\,,$$

the first term on the right is zero because X is Killing. By definition the Lie Derivative $\mathcal{L}_X$ acts on functions via $\mathcal{L}_X f := Xf$, furthermore on a vector field Y one has the identity $\mathcal{L}_X Y = [X, Y]$, so the equation above is equal to

$$Xg(Y, Z) = g([X, Y], Z) + g(Y, [X, Z])$$

which can be rewritten as

$$g(D_XY,Z)+g(Y,D_XZ)=g(D_XY-D_YX,Z)+g(Y,D_XZ-D_ZX)\,.$$

Subtracting common terms from both sides and rearranging yields the identity

$$g(Y,D_ZX)=-g(D_YX,Z)\,,$$

in particular with $Y=Z=\dot\gamma$ this tells us that the second term in (2.2.1) vanishes. So we see that $\langle\dot\gamma,X\rangle$ is constant along c. As it is zero at the point p the assertion follows. In this way we obtain an isometry $\pi\circ\gamma\colon I\to M/G$ (where I is equipped with its canonical metric and if necessary we may restrict I or translate the parameter, or identify endpoints to obtain the circle). We set

$$\phi\colon I\times G/K\to M\,,\quad \phi(t,gK)=g\cdot\gamma(t) \tag{2.2.2}$$

where K denotes the isotropy subgroup of p in G (it is equal to the isotropy subgroups $K_{\gamma(t)}$ whenever $\gamma(t)$ lies in a principal orbit, so ϕ is well defined). This map induces a diffeomorphism $\dot I\times G/K\to\bar M:=\phi\big(\dot I\times G/K\big)$, the image $\bar M$ being an open dense subset of M (here $\dot I$ denotes the interior of I). Moreover if we let G act on $I\times G/K$ by $g\cdot\big(t,\alpha K\big):=\big(t,g\alpha K\big)$ then ϕ turns into an G-equivariant map, hence in the third, fifth and sixth case of the table above ϕ is a global diffeomorphism. In the first case where M/G is a circle, all the orbits of G are principal orbits and the mapping $\pi\colon M\to M/G$ is a fibration that is locally trivialised with base a circle and fibre G/K. In all the remaining cases the dense open subset $\phi\big(\dot I\times G/K\big)$ of M is exactly the union of the principal orbits of M, the question that remains is what happens when we pass to the special orbits. Now if $p=c(0)$ correspond to a boundary point in M/G then the isotropy group H of p contains the principal isotropy group K and the coset space H/K identifies with the unit sphere in the subspace of T_pM normal to the orbit of p.

With this parametrisation in place one uses ϕ to pull back the metric on $\bar M$ to a metric dt^2+g_t on $\dot I\times G/K$ in order to conduct curvature computations and study their limit as we approach potential edges of I.

2.2.2 Formulae for sectional curvature

The metrics we study here are Riemannian submersions, enabling us to compute the curvature of the total space in terms of the curvature of the base and the fibre using O'Neill's T-tensor. These concepts are shortly explained here, following the exposition in [4]. Afterwards we apply the formalism in the particular cohomogeneity one setting of multiply warped products and write down sectional curvature formulae.

Preliminary material on Riemannian submersions

Let (M, g) and $(B, \tilde{g})$ be Riemannian manifolds and let $\mathcal{S}\colon M \to B$ be a submersion (i.e. at each point $p \in M$ the differential $\mathcal{S}_{*,p}\colon T_pM \to T_{\mathcal{S}(p)}B$ is surjective). The kernel of $\mathcal{S}_{*,p}$ is the tangent space to the fibre $F_b := \mathcal{S}^{-1}(p)$ (here $b := \mathcal{S}(p)$). It is called the *vertical subspace at* p and denoted by $\mathcal{V}_p$ and the orthogonal complement is called the *horizontal subspace at* p, denoted $\mathcal{H}_p$. The latter is identified with T_bB via the isomorphism of linear spaces

$$\mathcal{H}_p \xhookrightarrow{\iota} T_pM \xrightarrow{\mathcal{S}_{*,p}} T_bB \,. \tag{2.2.3}$$

We say that $\mathcal{S}$ is a *Riemannian submersion* if (at each point $p \in M$) we have

$$\mathcal{S}^*(\bar{g}) = g_{|\mathcal{H}} \,,$$

i.e. relative to the restriction $g_{|\mathcal{H}}$ of g to the horizontal subspace this map is an isometry.

A class of examples of Riemannian submersions arises as follows: take a Riemannian manifold $(I, \tilde{g})$, a smooth manifold N together with a family of metrics $\{g_t\}_{t\in I}$ on N parametrised by I, and consider the product manifold $I \times N$ with metric $g = \mathrm{proj}_1^*(\tilde{g}) + \mathrm{proj}_2^*(g_t)$ where $\mathrm{proj}_1^*, \mathrm{proj}_2^*$ denotes the pullback with respect to the canonical projection maps of the product $I \times N$. Then proj_1 is a Riemannian submersion.

Example: Cohomogeneity one metrics Of course the case where I is one - dimensional covers all those metrics on $I \times N$ of the form $dt^2 + g_t$ so, in particular, cohomogeneity one metrics fall into this class. Let us just point out the simplest cases below, the last example is the situation that we shall be concentrating on later.

- *Simple product*: if the family of metrics on N is independent of the parameter, so that $g_t = \bar{g}$ is constant in t, then we recover the usual product of two Riemannian manifolds $(I \times N, \tilde{g} + \bar{g})$ (in this case proj_2 is also a Riemannian submersion).

- *Warped product*: here the family of metrics on N is given by $g_t = f^2(t)\bar{g}$ where f is a smooth real valued function on I and $\bar{g}$ a fixed background metric on N.

- *Multiply warped product*: slightly more general is the case where N itself is a product, say $N = N_1 \times \cdots \times N_r$, and the fixed background metric on N arises as a sum $\bar{g} = \bar{g}_1 + \cdots + \bar{g}_r$ of fixed metrics on the factors, and the family is given by $g_t = f_1^2(t)\bar{g}_1 + \cdots f_r^2(t)\bar{g}_r$.

Suppose now that $\mathcal{S}\colon M \to B$ is a Riemannian submersion and let $\mathcal{T}(M)$ denote the space of smooth vector fields on M. We shall decompose a vector field $X \in \mathcal{T}(M)$ into its vertical and horizontal component relative to the submersion by writing $X = \mathcal{H}(X) + \mathcal{V}(X)$ where, pointwise for $p \in M$, the maps $\mathcal{H}$ and $\mathcal{V}$ are simply the orthogonal projections of T_pM onto $\mathcal{H}_p$ respectively $\mathcal{V}_p$. We say that $X \in \mathcal{T}(M)$ is *vertical* (respectively *horizontal*) if $\mathcal{H}(X) = 0$ (respectively $\mathcal{V}(X) = 0$). Further, let ∇ be the Levi - Civita connection of the metric g on M. For each $b \in B$ we denote by g_b the metric on the fibre F_b obtained by restricting g and ∇^b denotes the associated Levi - Civita connection. We would like to decompose the curvature of the space M in terms of the curvature of the spaces F_b and B. To this end Barrett O'Neill [34] introduced the so - called T*-tensor* $T\colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ defined by

$$T_X Y := \mathcal{H}\Big(\nabla_{\mathcal{V}(X)}\mathcal{V}(Y)\Big) + \mathcal{V}\Big(\nabla_{\mathcal{V}(X)}\mathcal{H}(Y)\Big). \tag{2.2.4}$$

This map is simply the second fundamental form of each fibre as one can see by applying it to vertical vector fields. There is an additional tensor (the A-tensor) that is needed to analyse the curvature of Riemannian submersions in general - however for cases where the base manifold B is one -dimensional this tensor vanishes so we shall not need it. We list and some of the properties of the T-tensor below:

Proposition 2.2.1. *Given vertical vector fields X, Y and a horizontal vector field H we have*

$$T_H X = 0 \qquad \textit{and} \qquad T_H H = 0 \tag{2.2.5}$$

$$T_X Y = \mathcal{H}(\nabla_X Y) \textit{ is horizontal} \tag{2.2.6}$$

$$T_X H = \mathcal{V}(\nabla_X H) \textit{ is vertical} \tag{2.2.7}$$

$$T_X Y = T_Y X \tag{2.2.8}$$

$$g\big(T_X Y, H\big) = -g\big(T_X H, Y\big) \tag{2.2.9}$$

Proof. The first three lines are immediate. For the fourth we recall that the second fundamental form is symmetric in its arguments. To see the final statement, note that

$$T_X Y = \nabla_X Y - \mathcal{V}(\nabla_X Y) \quad \text{and} \quad T_X H = \nabla_X H - \mathcal{H}(\nabla_X H)\,,$$

whilst the Levi - Civita connection satisfies the identity

$$X = g\big(\nabla_X Y, H\big) + g\big(Y, \nabla_X H\big)\,. \tag{2.2.10}$$

The left hand side in (2.2.10) vanishes since H and Y are orthogonal, hence

$$\begin{aligned} g\big(T_X Y, H\big) &= g\big(\nabla_X Y, H\big) - \underbrace{g\big(\mathcal{V}(\nabla_X Y), H\big)}_{=0} \overset{(2.2.10)}{=} -g\big(Y, \nabla_X H\big) \\ &= -g\big(T_X H, Y\big) - \underbrace{g\big(\mathcal{H}(\nabla_X H), Y\big)}_{=0}\,. \end{aligned}$$

□

Equipped with the T-tensor and the A - tensor O'Neill then writes down curvature formulae (sometimes called *O'Neill's curvature formulae*) that decompose the

Riemann curvature tensor

$$R(U,V)W = \nabla_{[U,V]}W - \nabla_U\nabla_V W + \nabla_V\nabla_U W \qquad \text{for } U,V,W \in TM\,.$$

on M. For our purposes the following special case is important:

Proposition 2.2.2 ([34], Theorem 1 and 3)**.** *Let $\mathcal{S}\colon M \to B$ be a Riemannian submersion of a Riemannian manifold (M,g) where B is one - dimensional. Let X, Y, Z, V be vertical vector fields and H, F horizontal vector fields as described above, further, let R_b denote the Riemannian curvature tensor of the metric g_b obtained by restricting the metric g to the fibre $\mathcal{S}^{-1}(b)$. Then*

$$\begin{aligned} g\big(R(X,Y)Z,V\big) &= g\big(R_b(X,Y)Z,V\big) - g\big(T_XZ,T_YV\big) + g\big(T_YZ,T_XV\big) \\ g\big(R(X,Y)Z,H\big) &= g\big((\nabla_YT)_XZ,H\big) - g\big((\nabla_XT)_YZ,H\big) \\ g\big(R(H,X)F,Y\big) &= g\big((\nabla_HT)_XY,F\big) - g\big(T_XH,T_YF\big)\,. \end{aligned} \tag{2.2.11}$$

This is all the material needed from the basic theory of Riemannian submersions. Before we move on to sectional curvature formulae for special cohomogeneity one manifolds let us mention that, instead of approaching curvature studies from the point of view of sectional curvature, one can of course also focus on the relationship of the Ricci curvature Ric and scalar curvature u of g to the Ricci curvature Ric_t respectively scalar curvature u_t of g_t. A proof for the next result can be found in [3, Proposition 3.11], it is essentially another application of the formulae given in Proposition 2.2.2 and holds for arbitrary cohomogeneity one metrics.

Proposition 2.2.3 ([3])**.** *Let $(X_i)_{i=1}^{n-1}$ be a local orthonormal basis for the space tangent to the factor M_0 in* (2.2.19)*. If X and Y are vertical then*

$$\mathrm{Ric}\,\big(X,Y\big) = \mathrm{Ric}_t\,\big(X,Y\big) - g\big(N,T_XY\big) + g\big((\nabla_HT)_XY,H\big) \tag{2.2.12}$$

$$\mathrm{Ric}\,\big(X,H\big) = g\big(\delta_tT(X),H\big) \tag{2.2.13}$$

$$\mathrm{Ric}\,\big(H,H\big) = Hg\big(N,H\big) - \|T\|^2 \tag{2.2.14}$$

$$u = u_t - \|N\|^2 = \|T\|^2 + 2Hg\big(N,H\big)\,. \tag{2.2.15}$$

where $N := \sum_i T_{X_i} X_i$ *is the mean curvature vector,* $\|T\|^2 = \sum_i \|T_{X_i} H\|^2$ *is the norm of* T *and* $\delta_t T(X) = -\sum_i (\nabla_{X_i} T)_{X_i} X$ *if* X *is vertical (all the expressions are independent of the chosen basis).*

The formulae above are useful for the study of complete cohomogeneity one Einstein manifolds. If we assume g to be Einstein, so that $\mathrm{Ric} = \lambda g$ for some constant λ, then the first three equations in Proposition 2.2.3 read

$$\mathrm{Ric}_t\big(X,Y\big) - g\big(N, T_X Y\big) + g\big((\nabla_H T)_X Y, H\big) = \lambda g\big(X,Y\big) \tag{2.2.16}$$

$$g\big(\delta_t T(X), H\big) = 0 \tag{2.2.17}$$

$$Hg\big(N,H\big) - \|T\|^2 = \lambda \tag{2.2.18}$$

Locally these equations always have a solution, hence $G/K \times (a,b)$ always admits G-invariant Einstein metrics where (a,b) is an interval. It is not, however, always possible to find such a metric that is also *complete* (i.e. the geodesics $\gamma(t)$ are defined for all $t \in \mathbb{R}$): in [3] L. Berard Bergery shows that a compact space G/K which admits an isotropy irreducible linear representation gives rise to an example where $G/K \times \mathbb{R}$ does not admit a complete G-invariant Einstein metric. (A homogeneous space $M = G/K$ is called *isotropy irreducible* if the action $\Psi \colon K \to Gl(T_pM)$, $\Psi(h)X = h_{*,p}(X)$ for $X \in T_pM$ is irreducible (here $h_{*,p}$ is the differential map induced by left translation by h). *Irreducible* in this context means that there is no proper invariant subspace, an *invariant subspace* is a linear subspace W of T_pM satisfying $\Psi(h)W \subset W$ for all $h \in K$.)

On the other hand, if one assumes completeness of an Einstein manifold then general results about Ricci curvature reduce the list of possible spaces. More concretely, if $\lambda > 0$ then Myers's theorem [32] implies that M is compact and has finite fundamental group, which means that the one - dimensional factor M/G cannot be the circle. For $\lambda \leq 0$ there is a result due to Bochner [6] which says that if M is a compact Riemannian manifold with non-positive Ricci curvature then all Killing fields X are parallel (A *Killing field* is a vector field whose integral flow induces diffeomorphisms that are isometries, that is the Lie Derivative $\mathcal{L}_X g$ vanishes. On the other hand, a vector field Y is said to be *parallel* if $\nabla Y = 0$.) In the cohomogeneity one setting this implies that the Riemann curvature tensor vanishes identically. Thus, if M is a compact Einstein manifold of cohomogeneity one then either its scalar curvature is strictly positive or sectional curvature of M vanishes everywhere. Finally, let us also mention that the splitting theorem by J. Cheeger and D. Gromoll [7] implies complete Ricci flat metrics that are irreducible must be either compact or the quotient space M/G must be isometric to $[0, \infty)$; in particular this tells us that there is a special orbit present in the space.

Sectional curvature formulae for multiply warped products

Let us now restrict considerations to multiply warped products as this is sufficient to study the sectional curvature of the concrete metrics considered in the following sections. In the case of multiply warped products the sectional curvature simplifies quite elegantly into separate terms that depend on the warping functions. So consider the total space

$$M = I \times M_0 = I \times (M_1 \times \cdots \times M_r) \tag{2.2.19}$$

where $(M_i, \bar{g}_i)$ is a Riemannian manifold of dimension d_i for each $i = 1, \ldots, r$. We endow M with a family of metrics

$$g = dt^2 + g_t \qquad \text{where} \qquad g_t = f_1^2(t)\bar{g}_1 + \cdots + f_r^2(t)\bar{g}_r \tag{2.2.20}$$

with $f_i > 0$ a given smooth function for each i. In this case the projection map $I \times M_0 \to I$ is a Riemannian submersion as described above.

Let us determine the T-tensor in terms of a local orthonormal frame of (M, g). Let $H = \partial/\partial t =: \partial_t$. For each $i = 1, \ldots, r$ let $\{\overline{Y}_{ij} \colon 1 \leq j \leq d_i\}$ be an orthonormal frame for $(M_i, \bar{g}_i)$, then

$$\{H, Y_{ij} = (1/f_i)\overline{Y}_{ij} \mid 1 \leq i \leq r \,, 1 \leq j \leq d_i\} \tag{2.2.21}$$

is an orthonormal frame for (M, g). As suggested in [3, §3.7] we choose the base $\overline{Y}_{ij}$ so that it commutes with H. In this case we have the following

Lemma 2.2.4. *If X, Y are vertical vector fields and H a horizontal vector field that commutes with X and Y then*

$$g\big(T_X Y, H\big) = -\frac{1}{2} H g\big(Y, X\big) \tag{2.2.22}$$

Remark 2.2.5. This identity is valid not only for multiply warped metrics but for cohomogeneity one metrics in general. It is also stated in [3, § 3.7].

Proof. We shall need the basic formula

$$\begin{aligned} g(\nabla_X Y, H) = \frac{1}{2}\Big[& X g(Y, H) + Y g(H, X) - H g(X, Y) + g([X, Y], H) \\ & - g([Y, H], X) - g([X, H], Y)\Big]\,. \end{aligned} \tag{2.2.23}$$

Now, using the Levi-Civita connection $\hat{\nabla}$ on the fibre we write

$$g\big(T_X Y, H\big) = g\big((\nabla_X Y - \hat{\nabla}_X Y), H\big) = g\big(\nabla_X Y, H\big)$$

where the last equality uses the orthogonality of $\hat{\nabla}_X Y$ to H. But this is then equal to

$$-\frac{1}{2} H g\big(X, Y\big) \tag{2.2.24}$$

where we have used the identity (2.2.23) together with the assumption that X and Y are chosen so as to commute with H ($[X, H] = [Y, H] = 0$), and finally that $[X, Y]$ is again a vertical vector field. To see the latter, choose local coordinates $(t, x^1, \ldots, x^n)$ and write

$$X = \sum_i a^i \partial_{x^i}\,, \qquad Y = \sum_j b^j \partial_{x^j}\,, \qquad [X, Y] = c^0 \partial_t + \sum_{l=1}^n c^l \partial_{x^l}\,.$$

Applying the bracket to the coordinate function t gives

$$c^0 = [X, Y](t) = \sum_{i=1}^n (a^i \frac{\partial t}{\partial x^i} - b^i \frac{\partial t}{\partial x^i}) = 0\,. \quad \square$$

From here we get the following identities:

Theorem 2.2.6 (Formulae for the T-tensor)**.** *Using the notation introduced in this section, we have*

$$T_H H = T_H \overline{Y}_{ij} = 0 \tag{2.2.25}$$

whilst

$$T_{\overline{Y}_{ij}} \overline{Y}_{kl} = \begin{cases} -(f_i f_i')H & \text{if } i = k \text{ and } j = l\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.26}$$

Finally,

$$T_{\overline{Y}_{ij}} H = \frac{1}{f_i} g\big(T_{\overline{Y}_{ij}} H, \overline{Y}_{ij}\big) Y_{ij} = \frac{f_i'}{f_i} \overline{Y}_{ij}\,. \tag{2.2.27}$$

Proof. From (2.2.6) we see that

$$T_{\overline{Y}_{ij}} \overline{Y}_{kl} = g\big(T_{\overline{Y}_{ij}} \overline{Y}_{kl}, H\big) H$$

is horizontal whereas (2.2.7) shows that

$$T_{\overline{Y}_{ij}} H = \sum_{k=1}^{r} \sum_{l=1}^{d_k} g\big(T_{\overline{Y}_{ij}} H, Y_{kl}\big) Y_{kl} \tag{2.2.28}$$

is vertical. Now from the identity in Lemma 2.2.4,

$$g\big(T_{\overline{Y}_{ij}} \overline{Y}_{kl}, H\big) = -\frac{1}{2} H g\big(\overline{Y}_{ij}, \overline{Y}_{kl}\big) = \begin{cases} -f_i' f_i & \text{if } i = k \text{ and } j = l\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

On the other hand, using (2.2.9) we see that

$$g\big(T_{\overline{Y}_{ij}} H, \overline{Y}_{kl}\big) = -g\big(T_{\overline{Y}_{ij}} \overline{Y}_{kl}, H\big)$$

and another application of Lemma 2.2.4 shows that this is equal to

$$= \frac{1}{2} H g\big(\overline{Y}_{ij}, \overline{Y}_{kl}\big) = \begin{cases} f_i' f_i & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise}\,. \end{cases}$$

Combining these results gives the identities (2.2.26) and (2.2.27) in the list of expressions for the T-tensor (the other follow immediately from Proposition 2.2.1). $\square$

Now that we have computed the T - tensor we can turn to sectional curvature. Sectional curvature of a Riemannian manifold M is a pointwise measure of curvature, denoted $K_p(X,Y)$ for $p \in M$ and (linearly independent) tangent vectors $X, Y \in T_pM$. The tangent vectors span a plane Π in T_pM, and sectional curvature is the Gaussian curvature of the surface obtained via the exponential map at p restricted to Π. In fact, this means that $K_p(X,Y)$ depends on X, Y only up to the plane that they span.

We shall use the following formula for sectional curvature in terms of the Riemann curvature tensor:

$$K(X,Y) = \frac{g\big(R(X,Y)X,Y\big)}{g\big(X,X\big)g\big(Y,Y\big) - g\big(X,Y\big)}\,, \tag{2.2.29}$$

if we choose X and Y to be orthogonal unit vectors then

$$K(X,Y) = g\big(R(X,Y)X,Y\big)\,. \tag{2.2.30}$$

Of course the sectional curvature may be computed once we know it on our basis vectors, that is $K(Y_{ij}, Y_{kl})$ respectively $K(Y_{ij}, H)$. The goal is to find formulae for these in terms of the sectional curvature K_t of the metrics g_t in the fibres, and it is here where the T-tensor becomes important, as can be seen from the following identities which are valid not only for multiply warped product metrics but more generally for cohomogeneity one metrics.

Proposition 2.2.7 ([4], Cor 9.29(a))**.** *Let (M,g) be the Riemannian manifold described in* (2.2.19)*, let K respectively K_t denote the sectional curvatures of the metrics g respectively g_t. Let H be a horizontal vector field and X,Y be vertical vector fields with $g(H,H) = g(X,X) = g(Y,Y) = 1$ and $g(X,Y) = 0$. Then*

$$K(X,Y) = K_t(X,Y) + g\big(T_XY, T_XY\big) - g\big(T_XX, T_YY\big) \tag{2.2.31}$$

$$K(H,X) = g\big((\nabla_H T)_X X, H\big) - g\big(T_XH, T_XH\big)\,. \tag{2.2.32}$$

Proof. The basic ingredient for the proof are O'Neill's Curvature formulae stated in Proposition 2.2.2. From the first line we see that with $Z = X$ and $V = Y$,

$$K(X,Y) = g\big(R(X,Y)X,Y\big) = g\big(R_t(X,Y)X,Y\big) - g\big(T_X X, T_Y Y\big) + g\big(T_Y X, T_X Y\big)$$
$$= K_t(X,Y) - g\big(T_X Y, T_X Y\big) + g\big(T_Y Y, T_X X\big)$$

where we used the symmetry (2.2.8) to obtain the last equality. Likewise, substituting $F = H$ and $Y = X$ into the third formula of (2.2.11) gives

$$K(H,X) = -g\big(R(H,X)H,X\big) = g\big((\nabla_H T)_X X, H\big) - g\big(T_X H, T_X H\big)$$

as required. $\square$

We are now ready to find the terms on the left hand side of (2.2.31) and (2.2.32) for our basis. Such formulae are stated for example in [12], we derive them here for our particular case.

Theorem 2.2.8 (Sectional curvature formulae)**.** *Given a multiply warped metric* (2.2.20) *and an orthonormal frame* (2.2.21)*, let Y_{ij}, Y_{kl} be any choice of distinct members. Then*

$$K(Y_{ij}, Y_{kl}) = K_t(\bar{Y}_{ij}, \bar{Y}_{kl}) - \frac{f_i' f_k'}{f_i f_k}\,. \tag{2.2.33}$$

whilst for any Y_{ij},

$$K(H, Y_{ij}) = -\frac{f_i''}{f_i}\,. \tag{2.2.34}$$

Proof. Equation (2.2.31) tells us that

$$K(Y_{ij}, Y_{ij'}) = K_t(Y_{ij}, Y_{ij'}) + g\big(T_{Y_{ij}} Y_{ij'}, T_{Y_{ij}} Y_{ij'}\big) - g\big(T_{Y_{ij}} Y_{ij}, T_{Y_{ij'}} Y_{ij'}\big)$$
$$= K_t(\bar{Y}_{ij}, \bar{Y}_{ij'}) + f_i^{-4}\Big(g\big(T_{\bar{Y}_{ij}} \bar{Y}_{ij'}, T_{\bar{Y}_{ij}} \bar{Y}_{ij'}\big) - g\big(T_{\bar{Y}_{ij}} \bar{Y}_{ij}, T_{\bar{Y}_{ij'}} \bar{Y}_{ij'}\big)\Big)$$

and the second term vanishes by if $j = j'$. If $j \neq j'$ then substituting (2.2.26) into the above gives

$$K(Y_{ij}, Y_{ij'}) = K_t(\bar{Y}_{ij}, \bar{Y}_{ij'}) - f_i^{-4}\Big(g\big(-(f_i f_i')H, -(f_i f_i')H\big)\Big)$$

$$= K_t(\bar{Y}_{ij}, \bar{Y}_{ij'}) - \left(\frac{f_i'}{f_i}\right)^2$$

On the other hand, if $i \neq k$ then

$$\begin{aligned}
K(Y_{ij}, Y_{kl}) &= K_t(Y_{ij}, Y_{kl}) + g\big(T_{Y_{ij}} Y_{kl}, T_{Y_{ij}} Y_{kl}\big) - g\big(T_{Y_{ij}} Y_{ij}, T_{Y_{kl}} Y_{kl}\big) \\
&= K_t(\bar{Y}_{ij}, \bar{Y}_{kl}) + (f_i f_k)^{-2}\Big(g\big(T_{\bar{Y}_{ij}} \bar{Y}_{kl}, T_{\bar{Y}_{ij}} \bar{Y}_{kl}\big) - g\big(T_{\bar{Y}_{ij}} \bar{Y}_{ij}, T_{\bar{Y}_{kl}} \bar{Y}_{kl}\big)\Big) \\
&\overset{(2.2.26)}{=} K_t(\bar{Y}_{ij}, \bar{Y}_{kl}) + (f_i f_k)^{-2}\Big(-g\big(-(f_i f_i')H, (-f_k f_k')H\big)\Big) \\
&= K_t(\bar{Y}_{ij}, \bar{Y}_{kl}) - \frac{f_i' f_k'}{f_i f_k}\,.
\end{aligned}$$

For the sectional curvature $K(H, Y_{ij})$, equation (2.2.32) tells us that

$$K(H, Y_{ij}) = f_i^{-2}\Big(g\big((\nabla_H T)_{\bar{Y}_{ij}} \bar{Y}_{ij}, H\big) - g\big(T_{\bar{Y}_{ij}} H, T_{\bar{Y}_{ij}} H\big)\Big) \tag{2.2.35}$$

whilst from the definition of ∇ on tensors we have

$$\begin{aligned}
g\big((\nabla_H T)_{\bar{Y}_{ij}} \bar{Y}_{ij}, H\big) = Hg\big(T_{\bar{Y}_{ij}} \bar{Y}_{ij}, H\big) - g\big(T_{\nabla_H \bar{Y}_{ij}} \bar{Y}_{ij}, H\big) - g\big(T_{\bar{Y}_{ij}} \nabla_H \bar{Y}_{ij}, H\big) \\
- g(T_{\bar{Y}_{ij}} \bar{Y}_{ij}, \nabla_H H\big)\,.
\end{aligned}$$

Note that $\nabla_H H = 0$. Also $\nabla_H \bar{Y}_{ij}$ is vertical, hence equation (2.2.8) says $T_{\nabla_H \bar{Y}_{ij}} \bar{Y}_{ij} = T_{\bar{Y}_{ij}} \nabla_H \bar{Y}_{ij}$. This simplifies the above expression to

$$g\big((\nabla_H T)_{\bar{Y}_{ij}} \bar{Y}_{ij}, H\big) = Hg\big(T_{\bar{Y}_{ij}} \bar{Y}_{ij}, H\big) - 2g\big(T_{\bar{Y}_{ij}} \nabla_H \bar{Y}_{ij}, H\big)$$

and (2.2.9) then gives

$$g\big((\nabla_H T)_{\bar{Y}_{ij}} \bar{Y}_{ij}, H\big) = Hg\big(T_{\bar{Y}_{ij}} \bar{Y}_{ij}, H\big) + 2g\big(T_{\bar{Y}_{ij}} H, \nabla_H \bar{Y}_{ij}\big)\,. \tag{2.2.36}$$

We recall that H and $\bar{Y}_{ij}$ commute, so $\nabla_H \bar{Y}_{ij} = \nabla_{\bar{Y}_{ij}} H$. Finally, $T_{\bar{Y}_{ij}} H$ is vertical, therefore

$$g\big(T_{\bar{Y}_{ij}} H, \nabla_{\bar{Y}_{ij}} H\big) = g\big(T_{\bar{Y}_{ij}} H, \mathcal{V}\big(\nabla_{\bar{Y}_{ij}} H\big)\big) = g\big(T_{\bar{Y}_{ij}} H, T_{\bar{Y}_{ij}} H\big)\,.$$

Applying the last two statements to (2.2.36) yields

$$g\big((\nabla_H T)_{\bar{Y}_{ij}} \bar{Y}_{ij}, H\big) = Hg\big(T_{\bar{Y}_{ij}} \bar{Y}_{ij}, H\big) + 2g\big(T_{\bar{Y}_{ij}} H, T_{\bar{Y}_{ij}} H\big)$$

and the formulae for the T-tensor in Theorem 2.2.6 imply that

$$g\big((\nabla_H T)_{\bar{Y}_{ij}}\bar{Y}_{ij}, H\big) = H(-f_i' f_i) + 2g\big(\frac{f_i'}{f_i}\bar{Y}_{ij}, \frac{f_i'}{f_i}\bar{Y}_{ij}\big)$$
$$= -f_i'' f_i - (f_i')^2 + 2(f_i')^2 = (f_i')^2 - f_i'' f_i \,. \tag{2.2.37}$$

Substituting this into (2.2.35) we obtain

$$K(H, Y_{ij}) = f_i^{-2}\Big(g\big((\nabla_H T)_{\bar{Y}_{ij}}\bar{Y}_{ij}, H\big) - g\big(T_{\bar{Y}_{ij}}H, T_{\bar{Y}_{ij}}H\big)\Big)$$
$$= f_i^{-2}\Big((f_i')^2 - f_i'' f_i - (f_i')^2\Big) = -\frac{f_i''}{f_i} \,. \tag{2.2.38}$$

□

We should also note that the product structure of g simplifies the situation due to the following Lemma, it will be the main ingredient to prove Proposition 2.2.10.

Lemma 2.2.9. *Let (M_i, g^i) for $i = 1, 2$ be Riemannian manifolds, with vector fields X_i respectively. We can view the X_i as vector fields on the product manifold $M_1 \times M_2$ with product metric $g_1 \oplus g_2$ (that is g_1 and g_2 are orthogonal). Then $\nabla_{X_i} X_j = 0$ for $i \neq j$, where ∇ denotes the Levi - Civita connection of the product metric.*

Proof. Let $\dim M_i = m_i$. Relative to local coordinates $(x^1, \dots, x^{m_1})$ for M_1 and $(x^{m_1+1}, \dots, x^{m_1+m_2})$ for M_2 the metric tensor is represented by the components

$$g_{ij} = \begin{cases} g^1_{ij} & \text{if } 1 \leq i, j \leq m_1 \\ g^2_{(i-m_1)(j-m_1)} & \text{if } m_1 < i, j \leq m_1 + m_2 \\ 0 & \text{otherwise} \,. \end{cases} \tag{2.2.39}$$

Since X_1 is tangent to M_1 we can represent it in the form $X_1 = \sum_{i=1}^{m_1} a_{1,i}\partial_{x^i}$ with $a_{1,i}$ independent of x^j for $m_1 < j \leq m_1 + m_2$ and similarly $X_2 = \sum_{i=m_1+1}^{m_1+m_2} a_{2,i}\partial_{x_i}$ with $a_{2,i}$ independent of x^j for $1 \leq j \leq m_1$. Now

$$\nabla_{X_1} X_2 = \sum_{i=1}^{m_1} \sum_{j=m_1+1}^{m_1+m_2} a_{1,i}(\partial_i a_{2,j})\partial_j + \sum_{i=1}^{m_1} \sum_{j=m_1+1}^{m_1+m_2} a_{1,i}a_{2,j}\nabla_{\partial_i}\partial_j \,. \tag{2.2.40}$$

The first term in (2.2.40) vanishes since $\partial_i a_{2,j} = 0$ in each summand. The second term vanishes because $\nabla_{\partial_i}\partial_j = 0$ in each summand. To see this, recall that

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{m_1+m_2} \Gamma_{ij}^k \partial_k \tag{2.2.41}$$

where

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{s=0}^{m_1+m_2} g^{ks}\Big(\underbrace{\partial_j g_{si}}_{(A)} + \underbrace{\partial_i g_{sj}}_{(B)} - \underbrace{\partial_s g_{ij}}_{(C)}\Big)\,. \tag{2.2.42}$$

The terms $(A) - (C)$ vanish identically. Indeed, from (2.2.39) we see that $g_{si} = g_{si}^1$ if $1 \leq s \leq m_1$. The left hand side is independent of x^j whenever $m_1 < j \leq m_1 + m_2$ and so $\partial_j g_{si} = 0$ in that case. Otherwise, for $m_1 < s \leq m_1 + m_2$ we have $g_{si} = 0$ hence $\partial_j g_{si} = 0$ and thus the terms labelled by (A) all vanish. The terms labelled by (B) vanish by a similar argument, finally the terms labelled with (C) vanish since $g_{ij} = 0$ whenever $1 \leq i \leq m_1$ and $m_1 < j \leq m_1 + m_2$.

This shows that the Christoffel symbols Γ_{ij}^k in (2.2.41) are all zero and therefore the claim holds. □

Proposition 2.2.10.

$$R(Y_{ij}, Y_{ij'})Y_{kl} = 0 \quad \textit{if } i \neq k\,. \tag{2.2.43}$$

Proof. By definition,

$$R\big(Y_{ij}, Y_{ij'}\big)Y_{kl} = \nabla_{Y_{ij}}\nabla_{Y_{ij'}}Y_{kl} - \nabla_{Y_{ij'}}\nabla_{Y_{ij}}Y_{kl} - \nabla_{[Y_{ij},Y_{ij'}]}Y_{kl}$$

Now Y_{ij}, $Y_{ij'}$ are tangent to the factor M_i, and so is $[Y_{ij}, Y_{ij'}]$ (see the proof of Lemma 2.2.4). On the other hand, Y_{kl} is tangent to the factor M_k and $i \neq k$, so the result follows from Lemma 2.2.9. □

The context above will suffice to conduct our study of sectional curvature asymptotics in Section 2.4.

2.3 Elementary function representation for the metrics found by Dancer and Wang

In this Section we establish elementary function representations for the metrics constructed by Dancer and Wang in [11]. They study the system of ODEs given by the cohomogeneity one Ricci - flat Einstein equation $\mathrm{Ric}\, g = 0$ as the Hamiltonian flow on the zero level set of a suitable Hamiltonian H and look for a function F such that

$$\{F, \mathsf{H}\} = \phi \mathsf{H} \tag{2.3.1}$$

for some function ϕ. In this case F is a conserved quantity on the zero level set of the Hamiltonian (as we have $\{F, \mathsf{H}\} = 0$) and, due to the low dimensionality of the phase space under consideration, this is enough to render the system integrable when restricted to the zero level set. For a large class of orbit types no non-trivial solutions exist, however in particular instances one *can* find functions F, ϕ that do not vanish identically, thereby making the system integrable on the zero level set of the Hamiltonian. From these cases Dancer and Wang construct new cohomogeneity one Einstein metrics. In the cohomogeneity one context, the treatment in [11] assumes that the principal orbit is a product $(G_1/K_1) \times (G_2/K_2)$ of distinct isotropy irreducible spaces which means that the metric is diagonal of the form

$$dt^2 + f_1^2(t)\bar{g}_1 + f_2^2(t)\bar{g}_2 \tag{2.3.2}$$

where $\bar{g}_i$ is a homogeneous background metric on the i^{th} component of the principal orbit. Setting $\mathrm{Ric}\, g = 0$, the equations (2.2.12)-(2.2.14) in Proposition 2.2.3 then reduce to the following system of non - linear ODEs:

$$\frac{f_1''}{f_1} + (d_1 - 1)\left(\frac{f_1'}{f_1}\right)^2 + d_2\frac{f_1' f_2'}{f_1 f_2} - \frac{A_1}{d_1 f_1^2} = 0 \tag{2.3.3}$$

$$\frac{f_2''}{f_2} + (d_2 - 1)\left(\frac{f_2'}{f_2}\right)^2 + d_1\frac{f_1' f_2'}{f_1 f_2} - \frac{A_2}{d_2 f_2^2} = 0 \tag{2.3.4}$$

$$d_1\frac{f_1''}{f_1} + d_2\frac{f_2''}{f_2} = 0 \tag{2.3.5}$$

where A_1, A_2 are non-zero constants, d_i denotes the dimension of the i^{th} component of the principal orbit, and we use a prime to denote differentiation with respect to t. In fact, the equations above may be derived in the presence of the metric from (2.3.2) alone when there may be no group action, with background metrics $\bar{g}_i$ that need not be homogeneous in general (though still Einstein).

Using the first two equations one can see that the third constraint is equivalent to

$$d_1(d_1-1)\left(\frac{f_1'}{f_1}\right)^2 + 2d_1d_2\frac{f_1'f_2'}{f_1f_2} + d_2(d_2-1)\left(\frac{f_2'}{f_2}\right)^2 - \frac{A_1}{f_1^2} - \frac{A_2}{f_2^2} = 0\,, \tag{2.3.6}$$

this provides the additional constraint $\mathsf{H} = 0$ to the Hamiltonian system (we refer to [9, 11] for details). For the dimension pairs $(d_1, d_2) = (2, 8), (3, 6)$ and $(5, 5)$, Dancer and Wang find that there exists a non-trivial solution to (2.3.1) which yields the Hamiltonian system integrable when restricted to the zero energy surface. The resulting cohomogeneity one Einstein metrics are defined by

$$f_2^{2\frac{d_1+1}{d_1-1}} = K\coth\left(\frac{R}{2}\right)\int \frac{\tanh\left(\frac{R}{2}\right)(\cosh R - 1)^{\frac{d_1+1}{d_1-1}}}{\sinh R}\, dR \tag{2.3.7}$$

and

$$f_1^{(d_1-1)}f_2^2 = \frac{C}{2A_1}(\cosh R - 1) \tag{2.3.8}$$

where $R = \sqrt{\frac{(d_1-1)A_1}{d_1}}r + \text{const.}$ depends on t via $r' := 1/f_1$ and C, K are non-zero constants.

Proposition 2.3.1. *For each of the stated dimension pair the following formulae for f_1 and f_2 satisfy* (2.3.8)-(2.3.7).

For $(d_1, d_2) = (2, 8)$:

$$f_2^6 = \frac{K}{4}\frac{30R - 16\sinh R + \sinh(2R) - 32\tanh(R/2) + 4E}{\tanh(R/2)} \tag{2.3.9}$$

$$f_1^3 = \frac{B(\cosh R - 1)^3\tanh(R/2)}{30R - 16\sinh R + \sinh(2R) - 32\tanh(R/2) + 4E} \tag{2.3.10}$$

where $B = \frac{C^3}{2A_1^3 K}$*, and* E *is a constant of integration.*

For $(d_1, d_2) = (3, 6)$*:*

$$f_2^4 = K\frac{\sinh R + 4\tanh(R/2) - 3R + E}{\tanh(R/2)} \tag{2.3.11}$$

$$f_1^4 = B\frac{(\cosh R - 1)^2 \tanh(R/2)}{\sinh R + 4\tanh(R/2) - 3R + E}\,. \tag{2.3.12}$$

where $B = \frac{C^2}{4A_1^2 K}$ *and* E *is a constant of integration.*

For $(d_1, d_2) = (5, 5)$*:*

$$f_2^3 = K\Big(\sqrt{2}\,\mathrm{csch}(R/2)\big(\cosh R + 3\big) + E\coth(R/2)\Big) \tag{2.3.13}$$

$$f_1^{12} = B\frac{(\cosh R - 1)^3}{\Big(\sqrt{2}\,\mathrm{csch}(R/2)\big(\cosh R + 3\big) + E\coth(R/2)\Big)^2} \tag{2.3.14}$$

where $B = \frac{C^3}{8K^2 A_1^3}$ *and* E *is a constant of integration.*

Proof. We treat the dimension pair separately and start with $(d_1, d_2) = (3, 6)$. In this case equation (2.3.7) simplifies to

$$f_2^4 = K\coth(R/2)\int \frac{\tanh(R/2)\left(\cosh R - 1\right)^2}{\sinh R}\,dR\,.$$

We have

$$\frac{\tanh\left(\frac{R}{2}\right)\left(\cosh R - 1\right)^2}{\sinh R} = \cosh R + \frac{4}{\cosh R + 1} - 3\,. \tag{2.3.15}$$

Indeed, since

$$\cosh R + \frac{4}{\cosh R + 1} - 3 = \frac{(\cosh R - 1)^2}{\cosh R + 1}$$

we only need to show that

$$\tanh(R/2)\,\mathrm{csch}\,R = (\cosh R + 1)^{-1}\,. \tag{2.3.16}$$

But this follows at once by expanding out the left hand using the identities $\cosh(2x) + 1 = 2\cosh^2 x$ and $\sinh(2x) = 2\sinh x\cosh x$.

Thus

$$\int \frac{\tanh\left(\frac{R}{2}\right)(\cosh R - 1)^2}{\sinh R}\, dR = \sinh R + 4\tanh(R/2) - 3R + \text{Const}$$

so that equation (2.3.11) is now clear. On the other hand, equation (2.3.8) simplifies to

$$f_1^2 f_2^2 = \frac{C}{2A_1}(\cosh R - 1) \ ,$$

hence $f_1^4 = \frac{C^2}{4A_1^2}(\cosh R - 1)^2 f_2^{-4}$, as required for equation (2.3.12).

Next we prove the claim associated to $(d_1, d_2) = (5, 5)$. In this case we have

$$f_1^4 f_2^2 = \frac{C}{2A_1}(\cosh R - 1) \tag{2.3.17}$$

and

$$f_2^3 = K \coth(R/2) \int \frac{\tanh(R/2)(\cosh R - 1)^{\frac{3}{2}}}{\sinh R}\, dR\,. \tag{2.3.18}$$

We rewrite

$$\frac{\tanh(R/2)(\cosh R - 1)^{\frac{3}{2}}}{\sinh R} = \frac{\sqrt{\cosh R - 1}(\cosh R - 1)}{\cosh R + 1} = \frac{\sinh R(\cosh R - 1)}{(\cosh R + 1)^{3/2}} \tag{2.3.19}$$

Changing variables to $u = \cosh R + 1$ this integrates to

$$\begin{aligned}\int \frac{\sinh R(\cosh R - 1)}{(\cosh R + 1)^{3/2}}\, dR &= 2(\cosh R + 1)^{1/2} + 4(\cosh R + 1)^{-1/2} + \text{Const} \\ &= \sqrt{2}\frac{(\cosh R + 3)}{\cosh(R/2)} + \text{Const}\end{aligned} \tag{2.3.20}$$

and multiplying this with $K\coth(R/2)$ gives (2.3.13) from which (2.3.14) is also clear.

Finally we deal with the case $(d_1, d_2) = (2, 8)$. Here equation (2.3.7) reduces to

$$f_2^6 = K \coth(R/2) \int \frac{\tanh(R/2)(\cosh R - 1)^3}{\sinh R}\, dR\,. \tag{2.3.21}$$

Using again (2.3.15) we see that

$$\int \frac{\tanh(R/2)\,(\cosh R-1)^3}{\sinh R}\,dR = \int \left(\cosh R + \frac{4}{\cosh R+1} - 3\right)(\cosh R - 1)\; dR$$
$$= \int \cosh^2 R + 4\frac{\cosh R-1}{\cosh R+1} - 4\cosh R + 3\,dR \tag{2.3.22}$$

and the latter expression has elementary anti - derivatives. Indeed, with

$$\int \cosh^2 R\,dR = \frac{1}{2}\big(R+\sinh(R)\cosh(R)\big) + \text{constant}$$

and

$$\int \frac{\cosh R-1}{\cosh R+1}\,dR = R - 2\tanh(R/2) + \text{constant}$$

the integral in (2.3.22) is

$$\begin{aligned}
&= \frac{1}{2}\big(R+\sinh(R)\cosh(R)\big) + 4\big(R-2\tanh(R/2)\big) - 4\sinh R + 3R + C\\
&= \frac{1}{2}\sinh(R)\cosh(R) - 8\tanh(R/2) - 4\sinh R + \frac{15}{2}R + C\\
&= \frac{1}{4}\left(\sinh(2R) - 32\tanh(R/2) - 16\sinh R + 30R\right) + C\,.
\end{aligned}$$

Substituting this into (2.3.21) yields (2.3.9). On the other hand, equation (2.3.7) reduces to

$$f_1 f_2^2 = \frac{C}{2A_1}\,(\cosh R - 1)\,,$$

hence $f_1^3 = \frac{C^3}{8A_1^3}\,(\cosh R-1)^3\,f_2^{-6}$ which is equation (2.3.10). □

2.4 Sectional curvature of the example metrics

The explicit solutions in Section 2.3 for f_1, f_2 are now studied in the context of the results from section 2.2.2. We take on the dimension pairs in turn.

2.4.1 The dimension pair $(2,8)$

Starting with $(d_1,d_2) = (2,8)$, we compute

$$\frac{d}{dt}f_1^3 = 3f_1^2 f_1' = 3f_1^2\left(\frac{df_1}{dR}\frac{dR}{dt}\right) = 3f_1^2\frac{df_1}{dR}\frac{\sqrt{A_1/2}}{f_1} = \frac{df_1^3}{dR}\frac{\sqrt{A_1/2}}{f_1} \tag{2.4.1}$$

so that

$$f_1' = \frac{df_1^3}{dR}\frac{\sqrt{A_1/18}}{f_1^3} = \frac{df_1^3}{dR}\sqrt{\frac{A_1}{18}}\frac{30R - 16\sinh R + \sinh(2R) - 32\tanh(R/2) + 4E}{B(\cosh R - 1)^3\tanh(R/2)} \tag{2.4.2}$$

where $B = \frac{C^2}{4A_1^2 K}$ and E is a constant of integration. From the r.h.s. of (2.3.10) we see that

$$\begin{aligned}\frac{df_1^3}{dR} =& \frac{4B\sinh^4(R/2)\tanh^2(R/2)(3\cosh R + 4)}{30R - 16\sinh R + \sinh(2R) - 32\tanh(R/2) + 4E} \\ &- \frac{64B\tanh(R/2)\sinh^8(R/2)\operatorname{csch}^2 R}{(30R - 16\sinh R + \sinh(2R) - 32\tanh(R/2) + 4E)^2}\end{aligned} \tag{2.4.3}$$

which asymptotically as $R \to \infty$ behaves like

$$\frac{4Be^{2R}(3e^R + 4)}{30R - 16e^R + e^{2R} - 32 + 4E} - \frac{64Be^{4R}e^{-2R}}{(30R - 16e^R + e^{2R} - 32 + 4E)^2} \backsim O(12Be^R)\,. \tag{2.4.4}$$

On the other hand, the third factor in (2.4.2), which is just f_1^{-3}, behaves for R large like

$$\frac{30R - 16e^R + e^{2R} - 32 + 4E}{Be^{3R}} \backsim O(\frac{1}{Be^R}) \tag{2.4.5}$$

and, recalling the fact that $t \to \infty$ as $R \to \infty$ it follows that

Proposition 2.4.1. *For the dimension pair* $(d_1, d_2) = (2, 8)$

$$f_1' \backsim O(1) \qquad \textit{as } t \to \infty\,. \tag{2.4.6}$$

We can also see from (2.4.5) that $f_1 \backsim O(e^{R/3})$, thus

Proposition 2.4.2. *For the dimension pair* $(d_1, d_2) = (2, 8)$,

$$\frac{f_1'}{f_1} \to 0 \qquad \textit{as } t \to \infty\,. \tag{2.4.7}$$

Next we turn to f_2 and its derivative. We have

$$\frac{d}{dt}f_2^6 = 6f_2^5 f_2' = 6f_2^5\left(\frac{df_2}{dR}\frac{dR}{dt}\right) = 6f_2^5\frac{df_2}{dR}\frac{\sqrt{A_1/2}}{f_1} = \frac{df_2^6}{dR}\frac{\sqrt{A_1/2}}{f_1}\,. \tag{2.4.8}$$

The r.h.s. of (2.3.9) yields $f_2^6 \backsim O(e^{2R})$ as $R \to \infty$, and we can also compute from it that

$$\frac{df_2^6}{dR} = \frac{K}{4}\frac{64\sinh^8(R/2)\operatorname{csch}^2 R}{\tanh(R/2)} - \frac{K}{8}\frac{30R - 16\sinh R + \sinh(2R) - 32\tanh(R/2) + 4E}{\sinh^2(R/2)} \tag{2.4.9}$$

(here K and E are constants). As $R \to \infty$ this behaves like

$$\frac{K}{4}\frac{64e^{4R}e^{-2R}}{1} - \frac{K}{8}\frac{30R - 16e^R + e^{2R} - 32 + 4E}{e^R} \backsim O(e^{2R}). \tag{2.4.10}$$

Using these asymptotics as well as $f_1 \backsim O(e^{R/3})$ as $R \to \infty$, and the fact that $t \to \infty$ if $R \to \infty$ we deduce

Proposition 2.4.3.

$$\frac{f_2'}{f_2} \overset{(2.4.8)}{=} \sqrt{\frac{A_1}{2}}\frac{df_2^6}{dR}\frac{1}{6f_2^6 f_1} \to 0 \qquad \textit{as } t \to \infty \tag{2.4.11}$$

This gives the asymptotics for sectional curvature $K(Y_{ij}, Y_{kl})$ determined by two vertical vectors in our base to be that of the fibre, we summarise this in a theorem.

Theorem 2.4.4. *For the dimension pair $(d_1, d_2) = (2, 8)$ sectional curvature of the plane spanned by pairs of basis vectors of the form $\{Y_{ij}, Y_{kl}\}$ is asymptotically given by the sectional curvature in the fibre, that is*

$$K\big(Y_{ij}, Y_{kl}\big) \to K_t\big(\bar{Y}_{ij}, \bar{Y}_{kl}\big) \quad \textit{as } t \to \infty\,. \tag{2.4.12}$$

Proof. From equation (2.2.33),

$$K(Y_{ij}, Y_{kl}) = K_t(\bar{Y}_{ij}, \bar{Y}_{kl}) - \frac{f_i' f_k'}{f_i f_k}$$

so the result follows directly from Proposition 2.4.2 and 2.4.3. $\square$

In order to determine the sectional curvature $K(H, Y_{ij})$ related to a plane spanned by a vertical vector H and the horizontal basis vector we need to determine second derivatives, for which we use the r.h.s. of (2.4.2) in the case of f_1. We have

$$\frac{d}{dt}f_1' = \frac{dR}{dt}\frac{d}{dR}\left(\frac{df_1^3}{dR}\sqrt{\frac{A_1}{18}}\frac{30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E}{B(\cosh R-1)^3\tanh(R/2)}\right)$$

$$= \frac{A_1}{6f_1}\left(\frac{d^2f_1^3}{dR^2}\frac{30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E}{B(\cosh R-1)^3\tanh(R/2)}\right.$$

$$\left. + \frac{df_1^3}{dR}\frac{d}{dR}\left\{\frac{30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E}{B(\cosh R-1)^3\tanh(R/2)}\right\}\right) \tag{2.4.13}$$

Looking at the term inside the brackets we already know that the second factor in the first summand is $O(e^{-R})$ and the first factor in the second summand is $O(e^R)$. For the remaining terms we compute

$$\frac{d^2f_1^3}{dR^2} = \sum_{i=1}^{4} F_i(R) \tag{2.4.14}$$

where

$$\begin{aligned}
F_1(R) &= \frac{\sinh^2(R/2)\tanh^3(R/2)\big(40\cosh R+9\cosh(2R)+35\big)}{4\big(30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E\big)} \\
&\backsim_{R\to\infty} \frac{e^R\big(40e^R+9e^{2R}+35\big)}{4\big(30R-16e^R+e^{2R}-32+4E\big)} = O(e^R) &(2.4.15)\\
F_2(R) &= -\frac{256B\sinh^{12}(R/2)\tanh^2(R/2)(3\cosh R+4)\operatorname{csch}^2 R}{(30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E)^2} \\
&\backsim_{R\to\infty} -\frac{256Be^{6R}(3e^R+4)e^{-2R}}{(30R-16e^R+e^{2R}-32+4E)^2} = O(-e^R) &(2.4.16)\\
F_3(R) &= -\frac{128B\sinh^{10}(R/2)\big(2\cosh R+5\big)\operatorname{csch}^4 R}{(30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E)^2} \\
&\backsim_{R\to\infty} -\frac{128Be^{5R}\big(2e^R+5\big)e^{-4R}}{(30R-16e^R+e^{2R}-32+4E)^2} = O(1) &(2.4.17)\\
F_4(R) &= \frac{8192B\tanh(R/2)\sinh^{16}(R/2)\operatorname{csch}^4 R}{(30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E)^3} \\
&\backsim_{R\to\infty} \frac{8192Be^{8R}e^{-4R}}{(30R-16e^R+e^{2R}-32+4E)^3} = O(e^{-2R}) &(2.4.18)
\end{aligned}$$

Hence

Proposition 2.4.5. *As $R\to\infty$,*

$$\frac{d^2f_1^3}{dR^2}\frac{30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E}{B(\cosh R-1)^3\tanh(R/2)} = O(1)\,. \tag{2.4.19}$$

For the second summand in (2.4.13) we still need to find the asymptotics of the second factor:

$$
\begin{aligned}
&\frac{d}{dR}\left\{\frac{30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E}{B(\cosh R-1)^3\tanh(R/2)}\right\}\\
&=\frac{64\sinh^8(R/2)\operatorname{csch}^2(R)}{B(\cosh R-1)^3\tanh(R/2)}\\
&-\frac{30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E}{\big(B(\cosh R-1)^3\tanh(R/2)\big)^2}\times\\
&\qquad\qquad\big(4B\sinh^4(R/2)\big(3\cosh R+4\big)\tanh^2(R/2)\big)\\
&\backsim_{R\to\infty}-\frac{30R-16e^R+e^{2R}-32+4E}{Be^{3R}}\big(4Be^{2R}\big(3e^R\big)\big)=O\big(-e^{2R}\big)\,. \qquad (2.4.20)
\end{aligned}
$$

and we see that

Proposition 2.4.6. *As* $R\to\infty$

$$
\frac{df_1^3}{dR}\frac{d}{dR}\left\{\frac{30R-16\sinh R+\sinh(2R)-32\tanh(R/2)+4E}{B(\cosh R-1)^3\tanh(R/2)}\right\}=O\big(-e^{3R}\big)\big) \qquad (2.4.21)
$$

Finally we need the asymptotics of f_1 which we know is $O\big(e^{R/3}\big)$. Using Proposition 2.4.5 and 2.4.6 we now find the asymptotic behaviour for (2.4.13) to be

$$
f_1''=O\big(-e^{8R/3}\big)\quad \text{as } R\to\infty\,. \qquad (2.4.22)
$$

so that, since $t\to\infty$ if $R\to\infty$,

Proposition 2.4.7.

$$
\frac{f_1''}{f_1}=O\big(-e^{7R/3}\big)\longrightarrow\infty\quad \text{as } t\to\infty\,. \qquad (2.4.23)
$$

This gives the first result for sectional curvature determined by planes spanned partially by the vector H:

Theorem 2.4.8. *For the dimension pair* $(d_1,d_2)=(2,8)$ *sectional curvature of the plane spanned by the pair* $\{H,Y_{1j}\}$ *for* $1\le j\le d_1$ *asymptotically tends towards infinity, that is*

$$
K\big(H,Y_{1j}\big)\to\infty\quad \text{as } t\to\infty\,. \qquad (2.4.24)
$$

Now for the analysis of f_2'' we start by reading from equation (2.4.8) that $f_2' = \frac{df_2^6}{dR}\frac{\sqrt{A_1/2}}{6f_2^5 f_1}$ and compute

$$f_2'' = \frac{d}{dt}f_2' = \frac{dR}{dt}\frac{d}{dR}\left(\frac{df_2^6}{dR}\frac{\sqrt{A_1/2}}{6f_2^5 f_1}\right) = \frac{A_1}{12f_1}\sum_{i=1}^{3} F_i(R) \tag{2.4.25}$$

where

$$F_1(R) = \frac{d^2 f_2^6}{dR^2}\frac{1}{f_1 f_2^5} = O\left(e^{2R}e^{-R/3}e^{-5R/3}\right) = O\big(1\big) \quad \text{as } R \to \infty \tag{2.4.26}$$

since $f_1 = O\big(e^{R/3}\big)$, $f_2^6 = O\big(e^{2R}\big)$ and, from (2.4.9),

$$\frac{d^2 f_2^6}{dR^2} = \sum_{j=1}^{4} G_j(R) \tag{2.4.27}$$

with

$$\begin{aligned}
G_1(R) &= \frac{K\left(16\sinh^2(R/2)\big(\cosh(R)+2\big)\tanh^3(R/2)\right)}{4\tanh(R/2)} \backsim_{R\to\infty} O\big(e^{2R}\big)\,,\\
G_2(R) &= -\frac{8K\sinh^8(R/2)\operatorname{csch}^2(R)}{\sinh^2(R/2)} \backsim_{R\to\infty} O\big(-e^R\big)\,,\\
G_3(R) &= -\frac{K64\sinh^8(R/2)\operatorname{csch}^2(R)}{8\sinh^2(R/2)} \backsim_{R\to\infty} O\big(-e^R\big)\,,\\
G_4(R) &= \frac{K\left(30R - 16\sinh(R) + \sinh(2R) - 32\tanh(R/2) + 4E\right)\sinh(R)}{2\sinh^4(R/2)}\\
&\qquad \backsim_{R\to\infty} \frac{K\left(30R - 16e^R + e^{2R} - 32 + 4E\right)e^R}{2e^{2R}} = O\big(e^R\big)\,.
\end{aligned}$$

Further, from (2.4.1) we see that $df_1/dR = \frac{1}{3f_1^2}\frac{df_1^3}{dR}$, and (2.4.4) tells that $df_1^3/dR = O\big(e^R\big)$ whilst $df_2^6/dR = O\big(e^{2R}\big)$ from (2.4.10). Hence

$$F_2(R) = -\frac{df_2^6}{dR}\frac{df_1/dR}{f_2^5 f_1^2} = -\frac{df_2^6}{dR}\frac{df_1^3/dR}{3f_2^5 f_1^4} = O\big(-e^{2R}\frac{e^R}{e^{5R/3}e^{4R/3}}\big) = O(1)\,. \tag{2.4.28}$$

Finally the last summand in the second factor of (2.4.25) is

$$F_3(R) = -5\frac{df_2^6}{dR}\frac{df_2/dR}{f_2^6 f_1} = -\frac{5}{6}\left(\frac{df_2^6}{dR}\right)^2\frac{1}{f_2^{11}f_1} = O\big(e^{4R}\frac{1}{e^{11R/3}e^{R/3}}\big) = O(1) \tag{2.4.29}$$

where we have used (2.4.8) to find $df_2/dR = \frac{1}{6f_2^5}\frac{df_2^6}{dR}$. We conclude that

Proposition 2.4.9.

$$f_2'' \to 0 \qquad as\ t \to \infty\,. \tag{2.4.30}$$

Proof. Substituting the asymptotics (2.4.26), (2.4.28) and (2.4.29) into the r.h.s of equation (2.4.25) shows that the second factor asymptotically behaves like a constant as $R \to \infty$. Since $f_1^{-1} = O(e^{-R/3})$ for R large and $t \to \infty$ if $R \to \infty$ the result follows. □

Theorem 2.4.10. *For the dimension pair $(d_1, d_2) = (2, 8)$ sectional curvature of the plane spanned by the pair $\{H, Y_{2j}\}$ for $1 \le j \le d_2$ asymptotically tends towards zero, that is*

$$K\big(H, Y_{2j}\big) \to 0 \quad as\ t \to \infty\,. \tag{2.4.31}$$

Proof. From (2.2.34),

$$K(H, Y_{2j}) = -\frac{f_2''}{f_2}\,.$$

The result is thus an immediate consequence of Proposition 2.4.9 since $f_2 = O(e^{R/3})$. □

2.4.2 The dimension pair $(3, 6)$

From (2.3.12) we obtain

$$\frac{d}{dt} f_1^4 = 4 f_1^3 f_1' = 4 f_1^3 \frac{df_1}{dR}\frac{dR}{dt} = \frac{df_1^4}{dR}\sqrt{\frac{2A_1}{3}}\frac{1}{f_1} \tag{2.4.32}$$

and the r.h.s of (2.3.12) shows that

$$\begin{aligned}\frac{df_1^4}{dR} = {} & \frac{8B\sinh^6(R/2)(2\cosh R + 3)\operatorname{csch}^2 R}{\sinh R + 4\tanh(R/2) - 3R + E} \\ & - \frac{B(\cosh R - 1)^2 \tanh(R/2)\big(\cosh R + 2\operatorname{sech}^2(R/2) - 3\big)}{\big(\sinh R + 4\tanh(R/2) - 3R + E\big)^2}\end{aligned} \tag{2.4.33}$$

which asymptotically as $R \to \infty$ behaves like

$$\frac{8B\big(e^{3R}/64\big)e^R 4e^{-2R}}{e^R/2} - \frac{B\big(e^{2R}/4\big)\big(e^R/2\big)}{\big(e^{2R}/4\big)} = O\big(e^R\big)\,. \tag{2.4.34}$$

Also, looking at (2.3.12) we see that $f_1^4 = O\big(e^R\big)$. Hence

$$f_1' = \sqrt{\frac{2A_1}{3}\frac{df_1^4}{dR}\frac{1}{4f_1^4}} = O\big(1\big) \tag{2.4.35}$$

and we conclude

Proposition 2.4.11.

$$\frac{f_1'}{f_1} \longrightarrow 0 \qquad \textit{as } t \to \infty\,. \tag{2.4.36}$$

Proof. Follows immediately from the asymptotics (2.4.35), $f_1 = O\big(e^{R/4}\big)$, and the fact that $t \to \infty$ if $R \to \infty$. □

Next we analyse f_2'. From (2.3.11) we obtain

$$\frac{d}{dt}f_2^4 = 4f_2^3 f_2' = 4f_2^3\frac{df_2}{dR}\frac{dR}{dt} = \frac{df_2^4}{dR}\sqrt{\frac{2A_1}{3}}\frac{1}{f_1} \tag{2.4.37}$$

whilst from the r.h.s. of the same equation one has

$$\begin{aligned}\frac{df_2^4}{dR} = {} & \frac{K\big(\cosh R + 2\,\mathrm{sech}^2(R/2) - 3\big)}{\tanh(R/2)} \\ & - \frac{K\big(\sinh R + 4\tanh(R/2) - 3R + E\big)\,\mathrm{sech}^2(R/2)}{2\tanh^2(R/2)}\end{aligned} \tag{2.4.38}$$

which, as $R \to \infty$, behaves like

$$K\big(\frac{1}{2}e^R + 8e^{-R} - 3\big) - \frac{K}{2}\big(\frac{1}{2}e^R + 4 - 3R + E\big)4e^{-R} = O\big(e^R\big) \tag{2.4.39}$$

so that

Proposition 2.4.12.

$$\frac{f_2'}{f_2} \longrightarrow 0 \qquad \textit{as } t \to \infty\,. \tag{2.4.40}$$

Proof. Equation (2.4.37) gives

$$\frac{f_2'}{f_2} = \sqrt{\frac{A_1}{24}}\frac{df_2^4}{dR}\frac{1}{f_1 f_2^4}\,,$$

hence $f_2'/f_2 = O\big(e^{-R/4}\big)$ from (2.4.39) and the asymptotics $f_1 = O\big(e^{R/4}\big), f_2^4 = O\big(e^R\big)$, hence the result follows since $t \to \infty$ if $R \to \infty$. □

Propositions 2.4.11 and 2.4.12 suffice to determine that sectional curvature of g induced by planes inside the fibres is asymptotically given by sectional curvature of the metric in the fibres, that is

Theorem 2.4.13. *For the dimension pair* $(d_1, d_2) = (3, 6)$ *sectional curvature of the plane spanned by pairs of basis vectors of the form* $\{Y_{ij}, Y_{kl}\}$ *is asymptotically given by the sectional curvature in the fibre:*

$$K(Y_{ij}, Y_{kl}) \to K_t(\bar{Y}_{ij}, \bar{Y}_{kl}) \qquad \text{as } t \to \infty\,. \tag{2.4.41}$$

Next we determine f_i'' for $i = 1, 2$. Starting with (2.4.32) it follows that

$$f_1'' = \sqrt{\frac{A_1}{24}} \frac{d}{dt} \left(\frac{df_1^4}{dR} \frac{1}{f_1^4} \right) = \sqrt{\frac{A_1}{24}} \frac{1}{f_1} \frac{d}{dR} \left(\frac{df_1^4}{dR} \frac{1}{f_1^4} \right) = \sqrt{\frac{A_1}{24}} \frac{1}{f_1} \sum_{i=1}^{3} H_i(R) \tag{2.4.42}$$

where

$$\begin{aligned} H_1(R) = {} & \frac{1}{f_1^4} \frac{B \tanh^3(R/2)\big(9\cosh R + 2\cosh(2R) + 9\big)}{\sinh R + 4\tanh(R/2) - 3R + E} \\ & - \frac{1}{f_1^4} \frac{8B \sinh^6(R/2)(2\cosh R + 3)\operatorname{csch}^2 R\big(\cosh R + 2\operatorname{sech}^2(R/2) - 3\big)}{\big(\sinh R + 4\tanh(R/2) - 3R + E\big)^2} \end{aligned}$$

which comes from the first term in (2.4.33). Since $f_1^4 = O\big(e^R\big)$, we see that the asymptotics of this term are

$$\frac{1}{e^R} \left(\frac{B\big(\frac{9}{2}e^R + e^{2R} + 9\big)}{e^R/2 + 4 - 3R + E} - \frac{8Be^{3R}/64(e^R+3)4e^{-2R}\big(e^R/2 + 8e^{-R} - 3\big)}{\big(e^R/2 + 4 - 3R + E\big)^2} \right) = O\big(1\big)\,, \tag{2.4.43}$$

next from the second term in (2.4.33) we obtain

$$\begin{aligned} H_2(R) = {} & -\frac{1}{f_1^4} \frac{192B \sinh^{12}(R/2)\big(\cosh R + 2\big)\operatorname{csch}^4 R}{\big(\sinh R + 4\tanh(R/2) - 3R + E\big)^2} \\ & + \frac{1}{f_1^4} \frac{2B(\cosh R - 1)^2 \tanh(R/2)\big(\cosh R + 2\operatorname{sech}^2(R/2) - 3\big)^2}{\big(\sinh R + 4\tanh(R/2) - 3R + E\big)^3} \end{aligned}$$

$$
\backsim_{R\to\infty} \frac{1}{e^R}\left(-\frac{192Be^{6R}/4096\big(e^R/2+2\big)16e^{-4R}}{\big(e^R/2+4-3R+E\big)^2} + \frac{2B(e^R/2-1)^2\big(e^R/2+8e^{-R}-3\big)^2}{\big(e^R/2+4-3R+E\big)^3} \right)
$$

$$
= O\big(1\big)\,, \tag{2.4.44}
$$

and finally

$$
H_3(R) = \frac{df_1^4}{dR}\frac{d}{dR}\left(\frac{1}{f_1^4}\right)\,,
$$

here we can use the r.h.s. of (2.3.12) and compute

$$
\begin{aligned}
\frac{d}{dR}\left(\frac{1}{f_1^4}\right) &= \frac{d}{dR}\left(\frac{\sinh R + 4\tanh(R/2) - 3R + E}{B(\cosh R - 1)^2\tanh(R/2)}\right)\\
&= \frac{\cosh R + 2\,\mathrm{sech}^2(R/2) - 3}{B(\cosh R - 1)^2\tanh(R/2)}\\
&\quad - \frac{\big(\sinh R + 4\tanh(R/2) - 3R + E\big)8\sinh^6(R/2)\big(2\cosh R + 3\big)\,\mathrm{csch}^2(R)}{B^2(\cosh R - 1)^4\tanh^2(R/2)}
\end{aligned}
$$

and as $R\to\infty$, this behaves like

$$
\frac{e^R/2+8e^{-R}-3}{Be^{2R}/4} - \frac{\big(e^R/2+4-3R+E\big)8e^{3R}/64e^R4e^{-2R}}{B^2e^{4R}/16} = O\big(e^{-R}\big)\,. \tag{2.4.45}
$$

Therefore

$$
H_3(R) = O\big(1\big) \tag{2.4.46}
$$

here we have also used (2.4.34) which tells us that $\frac{df_1^4}{dR} = O\big(e^R\big)$. We can now draw the conclusion that sectional curvature of planes spanned by base pairs of the form $\{H, Y_{1j}\}$:

Theorem 2.4.14. *For the dimension pair $(d_1, d_2) = (3, 6)$ sectional curvature of the plane spanned by the pair $\{H, Y_{1j}\}$ for $1 \le j \le d_1$ asymptotically tends towards zero, that is*

$$
K\big(H, Y_{1j}\big) \to 0 \qquad \text{as } t\to\infty\,. \tag{2.4.47}
$$

Proof. Recall from (2.2.34) that $K(H, Y_{1j}) = -\frac{f_1''}{f_1}$. Substituting (2.4.43), (2.4.44) and (2.4.46) into (2.4.42) and using the asymptotic behaviour $f_1 = O\big(e^{R/4}\big)$ we see that

$$\frac{f_1''}{f_1} \to 0 \qquad \text{as } R \to \infty$$

hence the result follows from the fact that $t \to \infty$ if $R \to \infty$. $\square$

Now for f_2''. We start with (2.4.37) from which we see that $f_2' = \sqrt{\frac{2A_1}{3}\frac{df_2^4}{dR}\frac{1}{4f_2^3 f_1}}$, hence

$$f_2'' = \frac{dR}{dt}\frac{d}{dR}f_2' = \frac{2A_1}{3}\frac{1}{4f_1}\frac{d}{dR}\left(\frac{df_2^4}{dR}\frac{1}{f_2^3 f_1}\right) = \frac{2A_1}{3}\frac{1}{4f_1}\sum_{i=1}^{3} A_i(R) \tag{2.4.48}$$

with

$$A_1(R) = \frac{1}{f_2^3 f_1}\frac{d}{dR}\frac{df_2^4}{dR} = \frac{1}{f_2^3 f_1}\sum_{j=1}^{4} F_j(R)$$

where (from the r.h.s. of (2.4.38)) we have

$$\begin{aligned}
F_1(R) &= \frac{K\big(\sinh R - 2\,\mathrm{sech}^2(R/2)\tanh(R/2)\big)}{\tanh(R/2)} =_{R\to\infty} O\big(e^R\big)\,, \\
F_2(R) &= -\frac{K\big(\cosh R + 2\,\mathrm{sech}^2(R/2) - 3\big)\,\mathrm{sech}^2(R/2)}{2\tanh^2(R/2)} =_{R\to\infty} O\big(1\big)\,, \\
F_3(R) &= \frac{-K\big(\cosh R + 2\,\mathrm{sech}^2(R/2) - 3\big)\,\mathrm{sech}^2(R/2)}{2\tanh^2(R/2)} \\
&\quad + \frac{K\big(\sinh R + 4\tanh(R/2) - 3R + E\big)\tanh(R/2)\,\mathrm{sech}^2(R/2)}{2\tanh^2(R/2)} =_{R\to\infty} O\big(Re^{-R}\big)\,, \\
F_4(R) &= \frac{K\big(\sinh R + 4\tanh(R/2) - 3R + E\big)\,\mathrm{sech}^4(R/2)}{2\tanh^3(R/2)} =_{R\to\infty} O\big(e^{-R}\big)\,.
\end{aligned} \tag{2.4.49}$$

Thus the term A_1 is asymptotically constant for R large, since $f_2^3 f_1 = O\big(e^R\big)$:

$$A_1(R) =_{R\to\infty} O\big(1\big)\,. \tag{2.4.50}$$

Further,

$$A_2(R) = \frac{df_2^4}{dR}\frac{1}{f_1}\frac{d}{dR}\frac{1}{f_2^3} = -3\frac{df_2^4}{dR}\frac{1}{f_1 f_2^4}\frac{df_2}{dR} = -\frac{3}{4}\left(\frac{df_2^4}{dR}\right)^2\frac{1}{f_1 f_2^7} \tag{2.4.51}$$

where we have used the identity $\dfrac{df_2^4}{dR} = 4f_2^3\dfrac{df_2}{dR}$ to obtain the last equality. Now from (2.4.39) we know that $\dfrac{df_2^4}{dR} = O\big(e^R\big)$, also $f_1 = O\big(e^{R/4}\big) = f_2$, hence

$$A_2(R) =_{R\to\infty} O\big(1\big)\,. \tag{2.4.52}$$

Similarly,

$$A_3(R) = \frac{df_2^4}{dR}\frac{1}{f_2^3}\frac{d}{dR}\frac{1}{f_1} = -\frac{df_2^4}{dR}\frac{1}{f_1^2f_2^3}\frac{df_1}{dR} = -\frac{df_2^4}{dR}\frac{df_1^4}{dR}\frac{1}{4f_1^5f_2^3} =_{R\to\infty} O\big(1\big) \tag{2.4.53}$$

since $\dfrac{df_1^4}{dR} = O\big(e^R\big)$ (see (2.4.35)). We are now ready to state the sectional curvature asymptotics determined by planes spanned by the horizontal basis vector and basis vectors of the second component to be zero:

Theorem 2.4.15. *For the dimension pair* $(d_1, d_2) = (3, 6)$ *sectional curvature of the plane spanned by the pair* $\{H, Y_{2j}\}$ *asymptotically vanishes, that is*

$$K\big(H, Y_{2j}\big) \to 0 \qquad \textit{as } t \to \infty\,. \tag{2.4.54}$$

Proof. Recall from theorem 2.2.8 that the sectional curvature formula for this case is $K\big(H, Y_{2j}\big) = -\dfrac{f_2''}{f_2}$. Using the asymptotics (2.4.50), (2.4.52) and (2.4.53) to analyse f_2'' for large t we see that $f_2'' \to 0$ as $t \to \infty$ (recall we use the property that $t \to \infty$ if $R \to \infty$). Since $1/f_2 = O\big(e^{-R/4}\big)$ the result follows. □

2.4.3 The dimension pair $(5, 5)$

For the last case $(d_1, d_2) = (5, 5)$, using (2.3.14) one has

$$\frac{d}{dt}f_1^{12} = 12f_1^{11}f_1' = 12f_1^{11}\left(\frac{df_1}{dR}\frac{dR}{dt}\right) = 12f_1^{11}\frac{df_1}{dR}\frac{\sqrt{4A_1/5}}{f_1} = 2\frac{df_1^{12}}{dR}\frac{\sqrt{A_1/5}}{f_1}\,, \tag{2.4.55}$$

hence

$$f_1' = \frac{df_1^{12}}{dR}\frac{\sqrt{A_1/5}}{6f_1^{12}} = \frac{df_1^{12}}{dR}\sqrt{\frac{A_1}{180}\frac{\Big(\sqrt{2}\,\mathrm{csch}(R/2)(\cosh R+3) + E\coth(R/2)\Big)^2}{B(\cosh R-1)^3}} \tag{2.4.56}$$

where $B = \frac{C^3}{8A_1^3K^2}$ and E is a constant of integration. Further, from the r.h.s. of (2.3.14) we calculate

$$\frac{df_1^{12}}{dR} = \frac{3\sinh R(\cosh R-1)^2}{\big(\sqrt{2}(\cosh R+3)\operatorname{csch}(R/2)+E\coth(R/2)\big)^2} + \frac{4\sinh^7(R/2)\big(9\sqrt{2}\cosh(R/2)-\sqrt{2}\cosh(3R/2)+2E\big)}{\big(E\cosh(R/2)+\sqrt{2}(\cosh R+3)\big)^3} \tag{2.4.57}$$

which asymptotically as $R\to\infty$ behaves as

$$\left(\frac{3(\frac{1}{4}-e^{-R}+e^{-2R})}{2\big(2\sqrt{2}(\frac{1}{2}+3e^{-R})+Ee^{-R/2}\big)^2}e^{2R} + \frac{\big(\frac{9}{2}\sqrt{2}e^{-R}-\frac{\sqrt{2}}{2}+2Ee^{-3R/2}\big)}{32\big(\frac{E}{2}e^{-R/2}+\sqrt{2}(\frac{1}{2}+3e^{-R})\big)^3}e^{2R}\right) \backsim O(\frac{1}{8}e^{2R}) \tag{2.4.58}$$

On the other hand, the third factor in (2.4.56) (which is f_1^{-12}) behaves for large R like

$$\frac{\Big(\sqrt{2}e^{R/2}+E\Big)^2}{B(e^R/2)^3} \backsim O(e^{-2R}) \tag{2.4.59}$$

and, recalling the fact that $t\to\infty$ as $R\to\infty$ it follows that

Proposition 2.4.16. *For the dimension pair $(d_1,d_2)=(5,5)$*

$$f_1' \backsim O(-1) \qquad \text{as } t\to\infty\,. \tag{2.4.60}$$

Moreover, (2.4.59) also tells us that $1/f_1 \backsim O(e^{-R/6})$, thus

Proposition 2.4.17. *For the dimension pair $(d_1,d_2)=(5,5)$,*

$$\frac{f_1'}{f_1} \to 0 \qquad \text{as } t\to\infty\,. \tag{2.4.61}$$

Next we consider f_2 and its derivative. We have

$$\frac{d}{dt}f_2^3 = 3f_2^2f_2' = 3f_2^2\left(\frac{df_2}{dR}\frac{dR}{dt}\right) = 3f_2^2\frac{df_2}{dR}\frac{\sqrt{4A_1/5}}{f_1} = \frac{df_2^3}{dR}\sqrt{\frac{4A_1}{5}}\frac{1}{f_1}\,. \tag{2.4.62}$$

The r.h.s. of (2.3.13) shows that $f_2^3 \backsim O(\sqrt{2}e^{R/2})$ as $R\to\infty$, and from it we can also compute

$$\frac{df_2^3}{dR} = K\Big(2\sqrt{2}\cosh(R/2) - \frac{\sqrt{2}}{2}\coth(R/2)\operatorname{csch}(R/2)\big(\cosh R+3\big) - \frac{E}{2}\operatorname{csch}^2(R/2)\Big) \tag{2.4.63}$$

(here K and E are constants). As $R \to \infty$ this behaves like

$$K\Big(\sqrt{2}e^{-R/2}e^{R} - \frac{\sqrt{2}}{2}e^{-R/2}e^{R} - \frac{E}{2}4e^{-R}\Big) \backsim O\big(K\frac{\sqrt{2}}{2}e^{R/2}\big) \tag{2.4.64}$$

Using these asymptotics as well as $1/f_1 \backsim O(\frac{B}{2}e^{-R/6})$ as $R \to \infty$, and the fact that $t \to \infty$ if $R \to \infty$, we deduce

Proposition 2.4.18.

$$\frac{f_2'}{f_2} \overset{(2.4.62)}{=} \frac{df_2^3}{dR}\sqrt{\frac{4A_1}{5}}\frac{1}{3f_2^3 f_1} \quad \backsim (Ke^{R/2})\sqrt{\frac{A_1}{5}}(\frac{B}{3}e^{-R/2}e^{-R/6}) \to 0 \quad \textit{as } t \to \infty\,. \tag{2.4.65}$$

This tells us that the asymptotics for sectional curvature $K(Y_{ij}, Y_{kl})$ determined by two *vertical* vectors in our base to be that of the fibre. We summarise this in a theorem:

Theorem 2.4.19. *For the dimension pair $(d_1, d_2) = (5, 5)$ sectional curvature of the plane spanned by pairs of basis vectors of the form $\{Y_{ij}, Y_{kl}\}$ is asymptotically given by the sectional curvature in the fibre, that is*

$$K\big(Y_{ij}, Y_{kl}\big) \to K_t\big(\bar{Y}_{ij}, \bar{Y}_{kl}\big) \quad \textit{as } t \to \infty\,. \tag{2.4.66}$$

Proof. From equation (2.2.33),

$$K(Y_{ij}, Y_{kl}) = K_t(\bar{Y}_{ij}, \bar{Y}_{kl}) - \frac{f_i' f_k'}{f_i f_k}$$

so the result follows directly from Proposition 2.4.17 and 2.4.18. □

In order to determine the sectional curvature $K(H, Y_{ij})$ related to a plane spanned by a vertical vector H and the horizontal basis vector we need to determine second derivatives; for this we use the r.h.s. of (2.4.56) in the case of f_1:

$$\frac{d}{dt}f_1' = \frac{dR}{dt}\frac{d}{dR}\left(\frac{df_1^{12}}{dR}\sqrt{\frac{A_1}{180}}\frac{\Big(\sqrt{2}\,\mathrm{csch}(R/2)(\cosh R + 3) + E\coth(R/2)\Big)^2}{B(\cosh R - 1)^3}\right)$$

$$= \frac{A_1}{15}\frac{1}{f_1}\Bigg(\frac{d^2 f_1^{12}}{dR^2}\frac{\Big(\sqrt{2}\,\mathrm{csch}(R/2)(\cosh R+3)+E\coth(R/2)\Big)^2}{B(\cosh R-1)^3}$$

$$+\frac{df_1^{12}}{dR}\frac{d}{dR}\Big\{\frac{\Big(\sqrt{2}\,\mathrm{csch}(R/2)(\cosh R+3)+E\coth(R/2)\Big)^2}{B(\cosh R-1)^3}\Big\}\Bigg) \tag{2.4.67}$$

Regarding the expression in brackets we already know from (2.4.59) that the second factor in the first summand is $O(e^{-2R})$ whilst from (2.4.58) we see that the first factor in the second summand is $O(e^{2R})$. For the remaining terms we compute

$$\frac{d^2 f_1^{12}}{dR^2} = \sum_{i=1}^{4} F_i(R) \tag{2.4.68}$$

where taking derivatives of the first summand in (2.4.57) results in the terms

$$\begin{aligned} F_1(R) &= \frac{12\sinh^4(R/2)(3\cosh R+2)}{\big(\sqrt{2}(\cosh R+3)\,\mathrm{csch}(R/2)+E\coth(R/2)\big)^2} \\ &\backsim_{R\to\infty} \frac{\frac{3}{4}e^{2R}(\frac{3}{2}+2e^{-R})}{\big(2\sqrt{2}(\frac{1}{2}+3e^{-R})+Ee^{-R/2}\big)^2} = O(\frac{9}{16}e^{2R}) \end{aligned} \tag{2.4.69}$$

$$\begin{aligned} F_2(R) &= -\frac{3\sinh R(\cosh R-1)^2\,\mathrm{csch}(R/2)}{\big(\sqrt{2}(\cosh R+3)\,\mathrm{csch}(R/2)+E\coth(R/2)\big)^3}\times \\ &\qquad \Big(2\sqrt{2}\sinh R-\sqrt{2}(\cosh R+3)\coth(R/2)-E\,\mathrm{csch}(R/2)\Big) \\ &\backsim_{R\to\infty} -\frac{3e^{2R}(\frac{1}{2}-e^{-R})^2\Big(\sqrt{2}-\sqrt{2}(\frac{1}{2}+3e^{-R})-2Ee^{-3R/2}\Big)}{\big(2\sqrt{2}(\frac{1}{2}+3e^{-R})+Ee^{-R/2}\big)^3} \\ &= O\big(-\frac{3}{16}e^{2R}\big) \end{aligned} \tag{2.4.70}$$

whilst taking derivatives of the second summand in (2.4.57) yields

$$\begin{aligned} F_3(R) &= \frac{2\sinh^6(R/2)(14E\cosh(R/2)+\sqrt{2}(34\cosh R-5\cosh(2R)+27))}{\big(E\cosh(R/2)+\sqrt{2}(\cosh R+3)\big)^3} \\ &\backsim_{R\to\infty} \frac{\frac{1}{32}e^{5R}(7Ee^{-3R/2}+\sqrt{2}(17e^{-R}-\frac{5}{2}+27e^{-2R}))}{e^{3R}\big(\frac{E}{2}e^{-R/2}+\sqrt{2}(\frac{1}{2}+3e^{-R})\big)^3} = O(-\frac{5}{16}e^{2R}) \end{aligned} \tag{2.4.71}$$

and

$$F_4(R) = -\frac{3(\frac{1}{2}E\sinh(R/2)+\sqrt{2}\sinh R)}{\big(E\cosh(R/2)+\sqrt{2}(\cosh R+3)\big)^4}\times$$

$$\Big(4\sinh^7(R/2)\big(9\sqrt{2}\cosh(R/2)-\sqrt{2}\cosh(3R/2)+2E\big)\Big)$$

$$\backsim_{R\to\infty} -\frac{\frac{3}{32}e^{6R}(\frac{1}{4}Ee^{-R/2}+\frac{\sqrt{2}}{2})(\frac{9}{2}\sqrt{2}e^{-R}-\frac{\sqrt{2}}{2}+2Ee^{-3R/2})}{e^{4R}\big(\frac{E}{2}e^{-R/2}+\sqrt{2}(\frac{1}{2}+3e^{-R})\big)^4} = O(\frac{3}{16}e^{2R}) \tag{2.4.72}$$

Hence

Proposition 2.4.20. *As* $R\to\infty$,

$$\frac{df_1^{12}}{dR}\sqrt{\frac{A_1}{180}}\frac{\Big(\sqrt{2}\operatorname{csch}(R/2)(\cosh R+3)+E\coth(R/2)\Big)^2}{B(\cosh R-1)^3} = O(1)\,. \tag{2.4.73}$$

For the second summand in (2.4.13) we also need to find the asymptotics of the second factor:

$$\frac{d}{dR}\Big\{\frac{\Big(\sqrt{2}\operatorname{csch}(R/2)(\cosh R+3)+E\coth(R/2)\Big)^2}{B(\cosh R-1)^3}\Big\}$$

$$= \frac{\sqrt{2}\big(E\coth(R/2)+\sqrt{2}(\cosh R+3)\operatorname{csch}(R/2)\big)}{B(\cosh R-1)^3}\times$$

$$\big(4\cosh(R/2)-\frac{1}{\sqrt{2}}E\operatorname{csch}^2(R/2)-(\cosh R+3)\coth(R/2)\operatorname{csch}(R/2)\big)$$

$$-\frac{3\sinh R\Big(\sqrt{2}\operatorname{csch}(R/2)(\cosh R+3)+E\coth(R/2)\Big)^2}{B(\cosh R-1)^4}$$

$$\backsim_{R\to\infty} \frac{2\sqrt{2}\big(Ee^{-R/2}+\sqrt{2}(\frac{1}{2}+3e^{-R})\big)}{Be^{2R}(\frac{1}{2}-e^{-R})^3}\times\big(2-2\sqrt{2}Ee^{-3R/2}-2(\frac{1}{2}+3e^{-R})\big)$$

$$-\frac{\frac{3}{2}\Big(2\sqrt{2}(\frac{1}{2}+3e^{-R})+Ee^{-R/2}\Big)^2}{Be^{2R}(\frac{1}{2}-e^{-R})^4} = O\big(-\frac{32}{B}e^{-2R}\big)\,. \tag{2.4.74}$$

Thus we see that

Proposition 2.4.21. *As* $R\to\infty$

$$\frac{df_1^{12}}{dR}\frac{d}{dR}\Big\{\frac{\Big(\sqrt{2}\operatorname{csch}(R/2)(\cosh R+3)+E\coth(R/2)\Big)^2}{B(\cosh R-1)^3}\Big\} = O\big(1\big) \tag{2.4.75}$$

Using Proposition 2.4.20 and 2.4.21 and the asymptotic behaviour of $1/f_1$ (which we know is $O\big(e^{-R/6}\big)$) we now find the asymptotic behaviour for (2.4.67) to be

$$f_1'' = O\big(e^{-R/6}\big) \quad \text{as } R\to\infty\,. \tag{2.4.76}$$

so that, since $t \to \infty$ if $R \to \infty$,

Proposition 2.4.22.

$$\frac{f_1''}{f_1} = O\big(e^{R/3}\big) \longrightarrow \infty \quad \textit{as } t \to \infty\,. \tag{2.4.77}$$

which gives the first result for sectional curvature determined by planes containing the vector H:

Theorem 2.4.23. *For the dimension pair $(d_1, d_2) = (5, 5)$ sectional curvature of the plane spanned by the pair $\{H, Y_{1j}\}$ for $1 \le j \le d_1$ asymptotically tends towards infinity, that is*

$$K\big(H, Y_{1j}\big) \to \infty \quad \textit{as } t \to \infty\,. \tag{2.4.78}$$

Now for the analysis of f_2'' we start with equation (2.4.62),

$$f_2' = \frac{df_2^3}{dR}\sqrt{\frac{4A_1}{5}}\frac{1}{2f_2^2 f_1}$$

and compute

$$f_2'' = \frac{d}{dt}f_2' = \frac{dR}{dt}\frac{d}{dR}\left(\frac{df_2^3}{dR}\sqrt{\frac{4A_1}{5}}\frac{1}{2f_2^2 f_1}\right) = \sqrt{\frac{A_1}{5}}\frac{1}{f_1}\sum_{i=1}^{3} F_i(R) \tag{2.4.79}$$

where

$$F_1(R) = \frac{d^2 f_2^3}{dR^2}\frac{1}{f_2^2 f_1} = O\big(1\big) \quad \text{as } R \to \infty \tag{2.4.80}$$

since $1/f_1 = O\big(e^{-R/6}\big)$, $1/f_2^2 = O\big(e^{-R/3}\big)$ and from (2.4.63),

$$\frac{d^2 f_2^3}{dR^2} = \frac{K}{16}\big(8E\cosh(R/2) + \sqrt{2}(4\cosh R + \cosh(2R) + 27)\big)\operatorname{csch}^3(R/2) \tag{2.4.81}$$

which for R large behaves like

$$\backsim_{R\to\infty} \frac{K}{16}\big(4Ee^{-3R/2} + \sqrt{2}(2e^{-R} + \frac{1}{2} + 27e^{-2R})\big)8e^{R/2} = O\big(\frac{\sqrt{2}K}{4}e^{R/2}\big)\,.$$

Further, from (2.4.55) we see that $\dfrac{df_1}{dR} = \dfrac{df_1^{12}}{dR}\dfrac{1}{6f_1^{11}}$, hence (2.4.64) and (2.4.58) show that

$$F_2(R) = -\frac{df_2^3}{dR}\frac{1}{f_2^2 f_1^2}\frac{df_1}{dR} = -\frac{1}{6}\frac{df_2^3}{dR}\frac{1}{f_2^2 f_1^{13}}\frac{df_1^{12}}{dR} = O\big(-e^{R/2}\frac{1}{e^{R/3}e^{13R/6}}e^{2R}\big) = O(1)\,. \tag{2.4.82}$$

Finally the last summand in the second factor of (2.4.79) is

$$F_3(R) = -2\frac{df_2^3}{dR}\frac{1}{f_2^3 f_1}\frac{df_2}{dR} = -\frac{2}{3}\left(\frac{df_2^3}{dR}\right)^2\frac{1}{f_2^5 f_1} = O\big(e^R\frac{1}{e^{5R/6}e^{R/6}}\big) = O(1) \tag{2.4.83}$$

where we have used (2.4.62) to find $\dfrac{df_2}{dR} = \dfrac{df_2^3}{dR}\dfrac{1}{3f_2^2}$. We conclude that

Proposition 2.4.24.

$$f_2'' \to 0 \qquad \text{as } t \to \infty\,. \tag{2.4.84}$$

Proof. Substituting the asymptotics (2.4.80), (2.4.82) and (2.4.83) into the r.h.s of equation (2.4.79) shows that the second factor asymptotically behaves like a constant as $R \to \infty$. Since $1/f_1 = O(e^{-R/6})$ for R large and $t \to \infty$ if $R \to \infty$ the result follows. □

Theorem 2.4.25. *For the dimension pair $(d_1, d_2) = (5,5)$ sectional curvature of the plane spanned by the pair $\{H, Y_{2j}\}$ for $1 \leq j \leq d_2$ asymptotically tends towards zero, that is*

$$K\big(H, Y_{2j}\big) \to 0 \quad \text{as } t \to \infty\,. \tag{2.4.85}$$

Proof. From (2.2.34),

$$K(H, Y_{2j}) = -\frac{f_2''}{f_2}\,.$$

The result is thus an immediate consequence of Proposition 2.4.80 since $1/f_2 = O(e^{-R/6})$. □

Chapter 3

A non-standard parametrix for the heat kernel on Riemannian manifolds with multiply warped metric

Given smooth compact Riemannian manifolds (M_1, g_1) and (M_2, g_2) we generalise the results of P.C. Lue in [29] and construct a parametrix for the fundamental solution to the heat equation on $I \times M_1 \times M_2$ with doubly warped metric $dr^2 + f_1^2(r)g_1 + f_2^2(r)g_2$. This gives rise to an asymptotic expansion for the heat trace in terms of the warping functions.

3.1 Introduction

Let $\mathcal{M}$ be a Riemannian manifold of dimension n. A continuous function

$$s \colon (0, \infty) \times \mathcal{M} \times \mathcal{M} \to \mathbb{C}\,, \quad (t, x, y) \mapsto s(t, x, y)$$

that is continuously differentiable in t and twice continuously differentiable in x and y such that

$$(\partial_t + \triangle_y)s = 0 \tag{3.1.1}$$

and with $\lim_{t\to 0} s(t,x,\cdot) = \delta_x$ (the Dirac - delta distribution based at $x \in \mathcal{M}$) is called a *fundamental solution* of the heat equation on $\mathcal{M}$, it is often referred to as a *heat kernel*. Here $\triangle_y$ denotes the Laplace - Beltrami operator with respect to the variable y, and the convergence means that for any smooth function ϕ with compact support on $\mathcal{M}$ the function $u(t,x) = \int_M s(t,x,y)\phi(y)d\mu(y)$ is continuous and $u(t,x) \to \phi(x)$ as $t \to 0_+$ where $d\mu$ denotes the Riemannian volume element on $\mathcal{M}$, locally given by $\sqrt{|g|}\,dx$ where $|g|$ denotes the determinant of the metric tensor and dx is Lebesgue measure in $\mathbb{R}^n$. A heat kernel exists and is unique, for example, on compact Riemannian manifolds without boundary. It is known explicitly for some manifolds, for example in the case where $\mathcal{M} = \mathbb{R}^n$ is Euclidean space endowed with the standard metric, we have $\triangle = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ in Cartesian coordinates, and

$$s(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\{-\frac{\|x-y\|^2}{4t}\}\,. \tag{3.1.2}$$

If $\mathcal{M}$ is compact of dimension n then there exists an asymptotic expansion [30] for s along the diagonal $y = x$,

$$s(t,x,x) \backsim_{t\to 0} \frac{1}{(4\pi)^{\frac{n}{2}}} \sum_{j\geq 0} a_j(x) t^{-\frac{n}{2}+j}\,. \tag{3.1.3}$$

The coefficients $a_j(x)$ depend on the curvature tensor R and its covariant derivatives, when integrated over the manifold $\mathcal{M}$ the resulting data can be interpreted in terms of the geometry of $\mathcal{M}$, for example $\int_{\mathcal{M}} a_0(x)dx$ is equal to the Riemannian volume of $\mathcal{M}$, and in the case where $\mathcal{M}$ is a surface $\int_{\mathcal{M}} a_1(x)dx = \pi\chi(M)/3$ with $\chi(M)$ the Euler characteristic of M. To obtain the asymptotic expansion (3.1.3) if suffices to construct an approximation to the heat kernel, also called a *parametrix*. Concretely one looks for a smooth function

$$p\colon (0,\infty) \times \mathcal{M} \times \mathcal{M} \to \mathbb{C}\,, \quad (t,x,y) \mapsto p(t,x,y) \tag{3.1.4}$$

such that $(\partial_t + \triangle)p$ extends to a continuous function on $[0,\infty) \times \mathcal{M} \times \mathcal{M}$ and such that $\lim_{t\to 0} p(t,x,\cdot) = \delta_x$ is the Dirac - delta distribution based at $x \in \mathcal{M}$ (a comparison of the defining conditions for $p(t,x,y)$ and $s(t,x,y)$ helps to appreciate the similarities of these two objects). There is standard procedure for this construction which was introduced (for the operator $\triangle - \partial_t$) by S. Minakshisundaram and A. Pleijel in [31], let us briefly recall the key steps (see for example [36, Chapter 3.2.1] for a more detailed exposition). One introduces Riemannian normal coordinates in a neighbourhood of a point $x \in M$, these are coordinates induced at a point x by the diffeomorphism $\exp_x \colon B_x(0,\varepsilon) \to U_x := \exp_x\big(B_x(0,\varepsilon)\big) \subset M$, $v \mapsto \gamma_v(1)$ where $\gamma_v \colon (-\delta,\delta) \to M$ is a geodesic that passes through x (i.e. $\gamma_v(0) = x$) and satisfies $\gamma_v'(0) = v$. (The freedom in the choice of ε for the size of the ball $B_x(0,\varepsilon)$ ensures that δ can be chosen > 1.) Let $\rho_x = \rho \colon U_x \to \mathbb{R}$, $y \mapsto \rho_x(y) = \rho(x,y)$ denote the length of the radial geodesic from x to y. To simplify the notation let us also denote

$$F_x(v) = F(v) := \det\big(\exp_x\big)_{*,v} \qquad \text{for } v \in B_x(0,\varepsilon). \tag{3.1.5}$$

Now, based on the Euclidean solution (3.1.2) one may argue that locally (i.e. in U_x or smaller if necessary) the heat kernel on M ought to be a perturbation of the function $(4\pi t)^{-n/2}\exp\{-\rho^2/(4t)\}$, so consider the sum

$$H_k(t,x,y) := \frac{1}{(4\pi t)^{n/2}} \exp\{-\frac{\rho_x^2(y)}{4t}\} \sum_{j=0}^{k} U_j(x,y) t^j\,. \tag{3.1.6}$$

If we then construct the functions $U_j(x,y)$ on the right hand side recursively as solutions to the differential equations $U_{-1} = 0$,

$$\rho \frac{\partial U_0}{\partial \rho} + \frac{\rho}{2} \frac{\partial F/\partial \rho}{F} U_0 = 0 \tag{3.1.7}$$

and

$$\rho \frac{\partial U_j}{\partial \rho} + \left(\frac{\rho}{2}\frac{\partial F/\partial \rho}{F} + i\right) U_j + \triangle_y U_{j-1} = 0 \qquad \text{for } j = 1, \ldots, k\,, \tag{3.1.8}$$

it follows that (3.1.6) satisfies the equation

$$(\partial_t + \triangle_y) H_k(\cdot, x, \cdot) = \frac{1}{(4\pi t)^{n/2}} \exp\{-\frac{\rho_x^2(y)}{4t}\} \triangle_y U_k(x,y) t^k\,.$$

That is, even though (3.1.6) is not an exact solution to the equation (3.1.1), the expression vanishes up to the highest power of t (i.e. up to the t^k term). Furthermore if one requires the U_i to be finite at $x = y$ and that $U_0(x,x) = 1$ (the normalising condition) then there are unique solutions

$$U_0(x,y) = \frac{1}{\sqrt{F(\exp_x^{-1}(y)}} \tag{3.1.9}$$

and for $j \geq 1$:

$$U_j(x,y) = \frac{1}{\rho_x^j(y)U_0(x,y)} \int_0^\rho \sqrt{F(v(s))}\, \triangle_y\, u_{j-1}\big(v(s),y\big) s^{j-1}\, ds\,. \tag{3.1.10}$$

Finally we extend H_k to $\mathcal{M} \times \mathcal{M}$ by choosing a bump function ψ with $\psi(s) \equiv 1$ for $s \leq R/2$ and $\psi(s) \equiv 0$ for $s > R$ where the constant $R > 0$ is small enough so that the geodesic ball $B_R(x)$ of radius R centered at x is contained in U_x for each $x \in \mathcal{M}$ (such an R exists uniformly in x in view of the assumption that $\mathcal{M}$ is compact and boundaryless). One can then prove that $p_k(t,x,y) := \psi\big(r(x,y)\big)H_k(t,x,y)$ yields a parametrix whenever $k > n/2$. Furthermore, one can see from the definition of the partial sums H_k in (3.1.6) that the result inherently yields an (asymptotic) expansion in t. This is the standard construction on compact manifolds without boundary.

Now let us consider a product manifold

$$M = I \times \mathcal{M}_1 \times \mathcal{M}_2 \quad \text{with metric} \quad dr^2 + f_1^2(r)g_1 + f_2^2(r)g_2 \tag{3.1.11}$$

where $(\mathcal{M}_i, g_i)$ for $i = 1,2$ are compact Riemannian manifolds, I is an open interval and the *warping* factors $f_1, f_2 \colon I \to (0,\infty)$ are smooth positive functions. Naturally one would like to know whether an expansion similar to (3.1.6) can be obtained (in case the heat kernel exists), and to what extend its coefficient functions $U_j\big((r,x),(r',y)\big)$ are determined in terms of the warping functions f_1, f_2 and the coefficients from the expansion (3.1.6) on $\mathcal{M}_1 \times \mathcal{M}_2$. This question was studied by P.C. Lue [29] for a *generalised surface of revolution* or warped product, that is a product manifold

$$M = I \times \mathcal{M} \quad \text{with warped metric} \quad dr^2 + f^2(r)g\,. \tag{3.1.12}$$

He remarks that this turns out to be very complicated if one uses the Ansatz (3.1.6) by making the following example calculation with $f(r) = r$ (in this this case (3.1.12) is sometimes called a *metric cone*): from (3.1.9) one sees that the first coefficient is the reciprocal of the square root of the determinant of the exponential map. Now for the metric cone one has

$$F_{(r,x)}\big(\exp^{-1}_{(r,x)}(r',y)\big) = \left(\frac{\rho_x(y)}{\sin\big(\rho_x(y)\big)}\right)^{n-1} F_x\big(\exp^{-1}_x(y)\big) \tag{3.1.13}$$

where n denotes the dimension of $\mathcal{M}$, and the functions F, exp on the left hand side are defined on M and on the right hand side they are the analogous objects defined on the base $\mathcal{M}$. Hence

$$U_0\big((r,x),(r',y)\big) = \left(\frac{\rho_x(y)}{\sin\big(\rho_x(y)\big)}\right)^{-(n-1)/2} U_0(x,y)$$

from which one can see that there is another factor coming from the base involving the distance function ρ on $\mathcal{M}$. The complexity that this term causes becomes apparent when one one starts to take recursively the Laplacian in (3.1.8) to determine the next coefficients. For cases more general than the metric cone the independence of the right hand side in (3.1.13) from r, r' cannot be taken for granted and the difficulty of the problem increases further. To get around this complication Lue suggests an alternative Ansatz. Using the eigenvalues and eigenfunctions $\{(\lambda_i, \phi_i)\}$ of the Laplacian on $\mathcal{M}$ he starts with the formal series

$$H_k\big(t,(r,x),(r',y)\big) = \frac{\big(f(r)f(r')\big)^{-n/2}}{(4\pi t)^{1/2}} \exp\{-\frac{(r-r')^2}{4t}\} \sum_{i\geq 0} \exp\{-\frac{\lambda_i t}{f(r)f(r')}\} \sum_{j=0}^{k} U_{ij}\big((r,x),(r',y)\big)t^j \tag{3.1.14}$$

where

$$U_{ij}\big((r,x),(r',y)\big) = a_j(r,r',\lambda_i)\phi_i(x)\phi_i(y)$$

and the functions a_j are to be determined. This provides, for any Riemannian manifold of the form (3.1.12), an asymptotic expansion of the heat kernel on M where the contributions of the warping function f and the contribution from the factor $\mathcal{M}$ is made more explicit.

The goal here is to show that this approach does not depend on the absence of additional warps. More concretely we show that it can be extended to Riemannian manifolds that are of the form (3.1.11). In Section 3.2 we lay out the formal series. Starting with a generic format we show that (at least formally) the most natural changes to Lue's guess (3.1.14) still work in the doubly warped case. The main result, shown in Section 3.3, is that the resulting parametrix as well as the essential features of the proof in [29] adapt to this case, and that the newly arising features are due to the fact that the coefficients a_j are now polynomials in more than one eigenvalue requiring some care so as to maintain the necessary estimates. The fact that we assume our metric (3.1.11) to be doubly warped (instead of multiply warped) is not important in the sense that the arguments below extend to multiply warped scenarios $I \times \mathcal{M}_1 \times \cdots \mathcal{M}_k$ with metric $dr^2 + f_1^2(r)g_1 + \cdots + f_k^2(r)g_k$.

3.2 The formal solution

Before we derive the form of the parametrix we point out some preliminary observations to be used later. Let $(\mathcal{M}_1^{d_1}, g_1)$ and $(\mathcal{M}_2^{d_2}, g_2)$ be compact Riemannian manifolds of dimension d_1, d_2 respectively and let I be an open interval. We shall study the Heat operator $\partial_t + \triangle$ on the manifold $M = I \times \mathcal{M}_1 \times \mathcal{M}_2$ with metric

$$dr^2 + f_1^2(r)g_1 + f_2^2(r)g_2\,. \tag{3.2.1}$$

where f_i for $i \in \{1,2\}$ is a smooth positive function $I \to (0,\infty)$. The scalar Laplace - Beltrami operator $\triangle$ on a Riemannian Manifold is given in local coordinates

$(x^1, \ldots, x^n)$ by

$$\triangle = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x_j} \right) , \tag{3.2.2}$$

here $\det g$ denotes the determinant of the matrix g representing the metric tensor locally, and $\left(g^{ij}\right)_{1 \leq i,j \leq n} = g^{-1}$ is the inverse, so that $g^{ik} g_{kj} = \delta^i_j$ with δ^i_j the Kronecker delta.

Proposition 3.2.1. *Let $\mathcal{M}$ be a smooth manifold, let I be an open interval. Given a smooth family of metrics g_r on $\mathcal{M}$ parametrised by $r \in I$, the scalar Laplace - Beltrami operator on $I \times \mathcal{M}$ with metric $dr^2 + g_r$ is given by*

$$\triangle = -\frac{\partial^2}{\partial r^2} - \frac{1}{2} \operatorname{tr}(g_r^{-1} \dot{g}_r) \frac{\partial}{\partial r} + \triangle_r \tag{3.2.3}$$

where $\triangle_r$ denotes the Laplace - Beltrami operator on $(\mathcal{M}, g_r)$ and $\dot{g}_r := \frac{\partial}{\partial r} g_r$.

Remark 3.2.2. The term $g_r^{-1}\dot{g}_r$ is the shape operator of the hypersurface $(\mathcal{M}, g_r)$ in M; its trace is the mean curvature of the hypersurface. (The shape operator L_r is a symmetric linear transformation on the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at p defined by $L_r X = \nabla_X H$ where ∇ denotes the Levi - Civita connection on M and H is the lift of a unit vector field from I to M, i.e. normal to the factor $\mathcal{M}$.)

Proof. The local matrix representing the warped metric $\tilde{g} = dr^2 + g_r$ is of the form

$$\tilde{g} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_{11}(r,x) & \cdots & g_{1n}(r,x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n1}(r,x) & \cdots & g_{nn}(r,x) \end{pmatrix} \quad \tilde{g}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & g^{11}(r,x) & \cdots & g^{1n}(r,x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g^{n1}(r,x) & \cdots & g^{nn}(r,x) \end{pmatrix} .$$

from which we can see that $\sqrt{\det \tilde{g}} = \sqrt{\det g_r}$, so

$$\begin{aligned} \triangle &= -\frac{1}{\sqrt{\det \tilde{g}}} \sum_{i,j=0}^{m} \frac{\partial}{\partial x_i} \left(\sqrt{\det \tilde{g}} \tilde{g}^{ij} \frac{\partial}{\partial x_j} \right) \\ &= -\frac{1}{\sqrt{\det g_r}} \frac{\partial}{\partial r} \left(\sqrt{\det g_r} \frac{\partial}{\partial r} \right) - \frac{1}{\sqrt{\det g_r}} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial}{\partial x_i} \left(\sqrt{\det g_r} g_r^{ij} \frac{\partial}{\partial x_j} \right) \end{aligned}$$

$$= -\frac{\partial^2}{\partial r^2} - \frac{1}{2}\operatorname{tr}(g_r^{-1}\dot{g}_r)\frac{\partial}{\partial r} + \triangle_r \tag{3.2.4}$$

where $\triangle_r$ denotes the Laplace - Beltrami operator on $(\mathcal{M}, g_r)$ and we used the identity $\frac{d}{dr}\det T_r = \det T_r \operatorname{tr}\left(T^{-1}\dot{T}_r\right)$ to obtain the second term in the last equation. □

Corollary 3.2.3. *The scalar Laplace - Beltrami operator $\triangle$ on the Riemannian manifold defined by* (3.1.11) *is given by*

$$\triangle = -\frac{\partial^2}{\partial r^2} - (d_1\frac{\dot{f}_1}{f_1} + d_2\frac{\dot{f}_2}{f_2})\frac{\partial}{\partial r} + \frac{1}{f_1^2}\triangle_{M_1} + \frac{1}{f_2^2}\triangle_{M_2} \tag{3.2.5}$$

with $\triangle_i$ the scalar Laplace - Beltrami operator on $(\mathcal{M}_i, g_i)$, for $i = 1, 2$.

Proof. This is an immediate application of the above result to the case $M = \mathcal{M}_1^{d_1} \times \mathcal{M}_2^{d_2}$ with $g_r = f_1^2(r)g_1 + f_2^2(r)g_2$. □

A further observation that we shall need concerns the decomposition of eigenfunctions for the Laplacian on product manifolds.

Proposition 3.2.4. *If φ is an eigenfunction with eigenvalue μ for $\triangle_1$ on $\mathcal{M}_1$ and if ψ is an eigenfunction with eigenvalue τ for $\triangle_2$ on $\mathcal{M}_2$ then*

$$\phi(x) = \phi(x_1, x_2) := \varphi(x_1)\psi(x_2)\,, \quad x = (x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2 \tag{3.2.6}$$

is an eigenfunction for $\triangle_1 + \triangle_2$ on $\mathcal{M}_1 \times \mathcal{M}_2$ with corresponding eigenvalue $\lambda = \mu + \tau$.

Proof. Applying $\triangle_1 + \triangle_2$ to ϕ immediately verifies the claim. □

Let us now turn to the formal derivation of the parametrix. Based on the Ansatz (3.1.14) we start with a formal double series of the form

$$\begin{aligned} &P(t, r, x, r', y) = \\ &\Psi(r, r')\exp\left(\frac{(r-r')^2}{kt}\right)\sum_{i=0}^{\infty}\exp\left(\frac{-\mu_i ct}{F_1(r, r')}\right)\exp\left(\frac{-\tau_i\tilde{c}t}{F_2(r, r')}\right)\cdot A_i(t, r, x, r', y) \end{aligned} \tag{3.2.7}$$

where F_1, F_2, Ψ are functions to be determined, $k, c, \tilde{c}$ are constants to be determined, and

$$A_i(t, r, x, r', y) = \sum_{j=0}^{\infty} a_j(r, r', \mu_i, \tau_i)\phi_i(x)\phi_i(y)t^{j-1/2} \tag{3.2.8}$$

with $0 \leq \mu_0 \leq \mu_1 \leq \cdots \nearrow \infty$ the eigenvalues of $\triangle_1$ on $(\mathcal{M}_1, g_1)$ and $0 \leq \tau_0 \leq \tau_1 \leq \cdots \nearrow \infty$ the eigenvalues of $\triangle_2$ on $(\mathcal{M}_2, g_2)$, lastly ϕ_i is the function defined in line (3.2.6).

Applying $\partial_t + \triangle_{(r',y)}$ to (3.2.7) yields

$$(\partial_t + \triangle)P = \sum_{i=0}^{\infty} \exp\left(\frac{(r-r')^2}{kt}\right) \exp\left(\frac{-\mu_i ct}{F_1}\right) \exp\left(\frac{-\tau_i \tilde{c}t}{F_2}\right) \cdot E_i \tag{3.2.9}$$

where E_i denotes the formal series

$$E_i(t, r, x, r', y) = \sum_{j=-2}^{\infty} e_j(r, r', \mu_i, \tau_i)\phi_i(x)\phi_i(y)t^{j-1/2} \tag{3.2.10}$$

whose coefficients e_j are linear combinations in $a_{j+2}, \ldots, a_{j-2}$ and their first and second derivatives (we set $a_k := 0$ whenever $k \leq 0$), concretely

$$e_{-2} = -\frac{k+4}{k^2}\,\Psi(r-r')^2 a_0\,, \tag{3.2.11}$$

$$\begin{aligned} e_{-1} = &-\frac{k+4}{k^2}\,\Psi(r-r')^2 a_1 \\ &+ \frac{4}{k}\Psi(r-r')\partial_{r'}a_0 + \frac{4}{k}\left(\partial_{r'}\Psi + \Theta\Psi\right)(r-r')a_0 - \frac{k+4}{2k}\,\Psi a_0 \end{aligned} \tag{3.2.12}$$

with $\Theta = \left(\frac{d_1}{2}\frac{\dot{f}_1}{f_1} + \frac{d_2}{2}\frac{\dot{f}_2}{f_2}\right)$, and for $j \geq 0$

$$\begin{aligned} e_j = &-\frac{4+k}{k^2}\,\Psi(r-r')^2 a_{j+2} \\ &+ \frac{4}{k}\Big[(r-r')\Psi\partial_{r'}a_{j+1} + \Big((r-r')\Theta\Psi + \frac{k}{4}(j+\frac{1}{2}-\frac{2}{k})\Psi + (r-r')\partial_{r'}\Psi\Big)a_{j+1}\Big] \\ &- \Psi\partial_{r'}^2 a_j - 2\left(\Theta\Psi + \partial_{r'}\Psi\right)\partial_{r'}a_j \\ &+ \Big[(\frac{\mu_i}{f_1^2} + \frac{\tau_i}{f_2^2})\Psi - \Phi\Psi - \dot{\Phi}\frac{4}{k}\Psi(r-r') - 2\Theta\partial_{r'}\Psi - \partial_{r'}^2\Psi\Big]a_j \\ &+ 2\dot{\Phi}\Psi\partial_{r'}a_{j-1} + \Big[2\dot{\Phi}\Theta\Psi + \ddot{\Phi}\Psi + 2\dot{\Phi}\partial_{r'}\Psi\Big]a_{j-1} \\ &- \dot{\Phi}^2\Psi a_{j-2} \end{aligned} \tag{3.2.13}$$

where $\Phi = \frac{c\mu_i}{F_1} + \frac{\tilde{c}\tau_i}{F_2}$ and $\dot{\Phi} = \partial_{r'}\Phi$ (for the arduous details of the derivation of the e_j we refer to Appendix 3.4.1). Now we set the e_j equal to zero. For $j = -2$ the equation is easily solved by setting

$$k = -4\,. \tag{3.2.14}$$

Next, if we let

$$\Psi(r, r') = \left(f_1(r) f_1(r')\right)^{-\frac{d_1}{2}} \cdot \left(f_2(r) f_2(r')\right)^{-\frac{d_2}{2}} \tag{3.2.15}$$

then the equation $e_{-1} = 0$ is satisfied when

$$a_0 = \text{constant}\,. \tag{3.2.16}$$

Let us also choose

$$F_1(r, r') = f_1(r) f_1(r') \quad \text{and} \quad F_2(r, r') = f_2(r) f_2(r')\,, \tag{3.2.17}$$

as well as

$$c = \tilde{c} = 1\,. \tag{3.2.18}$$

As a result of this we find

$$a_1(r, r', \mu_i, \tau_i) = \frac{a_0}{r - r'} \int_r^{r'} u_i(s, \mu_i, \tau_i)\, ds + a_0 \Phi \tag{3.2.19}$$

where

$$u_i = \Theta^2 + \dot{\Theta} + \left(\frac{\mu_i}{f_1^2} + \frac{\tau_i}{f_2^2}\right)\,, \tag{3.2.20}$$

furthermore

$$a_2(r, r', \mu_i, \tau_i) = \frac{a_0}{2(r - r')^2} \left(\int_r^{r'} u_i(s, \mu_i, \tau_i)\, ds\right)^2 + \frac{a_0}{(r - r')} \Phi \int_r^{r'} u_i(s, \mu_i, \tau_i)\, ds$$
$$- \frac{2a_0}{(r - r')^3} \int_r^{r'} u_i(s, \mu_i, \tau_i)\, ds - \frac{a_0}{(r - r')^2} \left[u_i(r, \mu_i, \tau_i) + u_i(r', \mu_i, \tau_i)\right] + \frac{a_0}{2} \Phi^2 \tag{3.2.21}$$

and for $j \geq 3$ we obtain

$$a_{j+1}(r,r',\mu_i,\tau_i) = \frac{1}{(r-r')^{j+1}} \int_r^{r'} \Big(-(r-s)^j \dot{\Phi}^2(r,s) a_{j-2}(r,s)$$
$$+ 2(r-s)^j \dot{\Phi}(r,s) \partial_s a_{j-1}(r,s) + (r-s)^j \ddot{\Phi}(r,s) a_{j-1}(r,s) - (r-s)^j \partial_s^2 a_j(r,s)$$
$$+ (r-s)^j \big[u_i(s) - \Phi(r,s)\big] a_j(r,s) + (r-s)^{j+1} \dot{\Phi}(r,s) a_j(r,s) \Big)\, ds\,. \tag{3.2.22}$$

Remark 3.2.5. The dependence of the terms on the l.h.s on the eigenvalues μ_i, τ_i has been suppressed since it is not relevant for the integration.

Remark 3.2.6. For brevity we have only stated the final expressions in (3.2.19) - (3.2.22) (details of the derivation are given in Section 3.4.2).

Let us finish this section by establishing two specific properties of the a_j which are needed later. The first concerns their smoothness and the second addresses their polynomial degrees in μ_i and τ_i.

Lemma 3.2.7. *For each $j \geq 0$ and all $i \geq 0$ the coefficient function $a_j(\cdot,\cdot,\mu_i,\tau_i)$ is C^∞ in r and r'.*

Proof. We use the inductive argument presented in [29]. The statement is certainly true for $j = 0$ since a_0 is a constant. Suppose $a_k(\cdot,\cdot,\mu_i,\tau_i)$ is smooth for $0 \leq k \leq j$. From (3.2.22) we see that

$$a_{j+1}(r,r',\mu_i,\tau_i) = \frac{1}{(r-r')^{j+1}} \int_r^{r'} (r-s)^j F(r,s)\, ds$$

where

$$F(r,s) = -\dot{\Phi}^2(r,s) a_{j-2}(r,s) + 2\dot{\Phi}(r,s) \partial_s a_{j-1}(r,s) + \ddot{\Phi}(r,s) a_{j-1}(r,s)$$
$$- \partial_s^2 a_j(r,s) + \big[u_i(s) - \Phi(r,s)\big] a_j(r,s) + (r-s)\dot{\Phi}(r,s) a_j(r,s)\,.$$

This function is smooth in both arguments since it is the sum of products involving the smooth functions Φ (c.f. (3.4.25)), u (c.f. (3.4.28) and (3.4.26)) and their derivatives, as well as a_k for $0 \leq k \leq j$ and derivatives thereof (these are smooth by the inductive hypothesis). This establishes that a_{j+1} is C^∞ in r and r' whenever $r \neq r'$. For the case $r = r'$ we use the Taylor expansion of F in the second variable at $s = r$,

$$\begin{aligned}
a_{j+1}(r,r') = \frac{(-1)^j}{(r-r')^{j+1}} \Bigg\{ & F(r,r) \int_r^{r'} (s-r)^j \, ds + \frac{\partial_s F(r,r)}{1!} \int_r^{r'} (s-r)^{j+1} \, ds + \\
& \cdots + \frac{1}{(\alpha+\beta)!} \int_r^{r'} \big(\partial_s^{\alpha+\beta} F\big)(r,\tau_s)(s-r)^{j+\alpha+\beta} \, ds \Bigg\} \\
= \frac{(-1)^j}{(r-r')^{j+1}} \Bigg\{ & \frac{F(r,r)}{j+1}(r'-r)^{j+1} + \frac{\partial_s F(r,r)}{1!(j+2)}(r'-r)^{j+2} + \\
& \cdots + \frac{1}{(\alpha+\beta)!} \int_r^{r'} \big(\partial_s^{\alpha+\beta} F\big)(r,\tau_s)(s-r)^{j+\alpha+\beta} \, ds \Bigg\} \\
= -\frac{F(r,r)}{j+1} & - \frac{\partial_s F(r,r)}{1!(j+2)}(r'-r) - \\
& \cdots - \frac{1}{(\alpha+\beta)!(r'-r)^{j+1}} \int_r^{r'} \big(\partial_s^{\alpha+\beta} F\big)(r,\tau_s)(s-r)^{j+\alpha+\beta} \, ds
\end{aligned} \tag{3.2.23}$$

where $r < \tau_s < s$. The only term that is not obviously smooth when $r = r'$ is the last, so let us look at

$$\frac{\partial^{\gamma+\delta}}{\partial r^\gamma \partial (r')^\delta} \left(\frac{1}{(r'-r)^{j+1}} \int_r^{r'} \big(\partial_s^{\alpha+\beta} F\big)(r,\tau_s)(s-r)^{j+\alpha+\beta} \, ds \right) . \tag{3.2.24}$$

When we apply the derivatives the result is going to be a sum where a generic term is a linear combination of

$$\frac{\text{constant}}{(r'-r)^{j+1+k}} \int_r^{r'} \big(\partial_s^{\alpha+\beta} F\big)(r,\tau_s)(s-r)^{j+\alpha+\beta} \, ds \tag{3.2.25}$$

(with $k < \gamma + \delta$) and

$$\frac{1}{(r'-r)^{j+1+k}} G(r,r')(r'-r)^{j+\alpha+\beta-k'} \tag{3.2.26}$$

where $k < k'$, and $G(r,r')$ is a derivative of $\big(\partial_{r'}^{\alpha+\beta} F\big)(r,\tau_{r'})$ therefore smooth in r, r'. Note that the number of derivatives applied in (3.2.26) to the polynomial factor in the numerator and denominator is obviously bounded by the total number of derivatives, that is $k + k' \leq \gamma + \delta$. Thus for both (3.2.25) and (3.2.26) the limit exists as $r' \to r$ since we may chose $\alpha + \beta$ as large as needed. In fact since $k' \leq \gamma + \delta - k$ it follows that

$$(j + \alpha + \beta - k') - (j + 1 + k) \geq \alpha + \beta - (\gamma + \delta + 1)$$

which is positive provided $\alpha + \beta > \gamma + \delta + 1$. Thus we see that (3.2.24) exists at $r = r'$ for any choice of γ, δ, in other words the last term in the Taylor expansion (3.2.23) is smooth as well. □

As mentioned before we need one further property of the a_j, namely their degree as polynomials in the eigenvalues μ_i and τ_i. This plays a role in the proof of Lemma 3.3.2 where we study the continuity of the Parametrix near $t = 0$.

Lemma 3.2.8. *The degree, denoted by $d(\cdot)$, of a_j and its derivatives at $r' = r$, seen as polynomials in the eigenvalues μ_i, τ_i, satisfies the following bounds:*

1. $d\big(a_j(r, r, \mu_i, \tau_i)\big) \leq \lceil 2j/3 \rceil$.
2. $d\left(\left(\partial_{r'}^k a_j(r, r', \mu_i, \tau_i a)\right)_{r'=r}\right) \begin{cases} \leq \lceil (2j + k)/3 \rceil & \text{for } 0 < k \leq j\,, \\ \leq j & \text{for } k \geq j\,. \end{cases}$

where $\lceil \cdot \rceil$ denotes the ceiling function.

Remark 3.2.9. This property departs from the analogue in [29] as we were not able to establish the sharper bound $d\left(\left(\partial_{r'}^k a_j(r, r', \mu_i, \tau_i a)\right)_{r'=r}\right) \leq \lceil (2j + k - 1)/3 \rceil$ for $0 < k \leq j$ stated there.

Proof for Propery 1. Proceeding by induction, for the base case we use (3.2.19) to compute

$$a_1(r, r, \mu_i, \tau_i) = \lim_{r \to r'} \left(\frac{a_0}{r - r'} \int_r^{r'} \Theta^2 + \dot{\Theta}\, ds \right)$$

$$+\lim_{r'\to r}\Big(\frac{a_0}{r-r'}\int_r^{r'}\frac{\mu_i}{f_1(s)^2}+\frac{\tau_i}{f_2^2(s)}\,ds\Big)+a_0\Big(\frac{\mu_i}{f_1(r)^2}+\frac{\tau_i}{f_2(r)^2}\Big) \tag{3.2.27}$$

The first term has zero degree since Θ does (c.f. the line below equation (3.2.12) for Θ). For the terms in the second line note that the function

$$I(r')=\int_r^{r'}\frac{\mu_i}{f_1(s)^2}+\frac{\tau_i}{f_2^2(s)}\,ds \tag{3.2.28}$$

is differentiable and

$$\begin{aligned}&\lim_{r'\to r}\Big(\frac{1}{r-r'}\int_r^{r'}\frac{\mu_i}{f_1(s)^2}+\frac{\tau_i}{f_2^2(s)}\,ds\Big)\\ &=-\lim_{r'\to r}\frac{I(r')-I(r)}{r'-r}=-I'(r)=-\Big(\frac{\mu_i}{f_1(r)^2}+\frac{\tau_i}{f_2^2(r)}\Big)\,.\end{aligned} \tag{3.2.29}$$

In other words, the last line cancels out so that $a_1(r,r,\mu_i,\lambda_i)$ has degree zero and the statement is true for $j=1$. Assuming it holds for all $j'\le j$ we need to show it is satisfied by

$$\begin{aligned}a_{j+1}(r,r',\mu_i,\tau_i)&=\frac{1}{(r-r')^{j+1}}\int_r^{r'}\Bigg(-(r-s)^j\dot\Phi^2(r,s)a_{j-2}(r,s)\\ &+2(r-s)^j\dot\Phi(r,s)\partial_s a_{j-1}(r,s)+(r-s)^j\ddot\Phi(r,s)a_{j-1}(r,s)-(r-s)^j\partial_s^2 a_j(r,s)\\ &+(r-s)^j\big[u_i(s)-\Phi(r,s)\big]a_j(r,s)+(r-s)^{j+1}\dot\Phi(r,s)a_j(r,s)\Bigg)\,ds\,.\end{aligned} \tag{3.2.30}$$

Here, the factor

$$\Phi=\mu_i/\big(f_1(r)f_1(r')\big)+\tau_i/\big(f_2(r)f_2(r')\big) \tag{3.2.31}$$

has degree 1 and so does u_i since Θ has degree zero (c.f. equation (3.2.20) for u_i). Now consider

$$\begin{aligned}I(r')&:=(r-r')^{j+1}a_{j+1}(r,r',\mu_i,\tau_i)\\ &=\int_r^{r'}-(r-s)^j\dot\Phi^2(r,s)a_{j-2}(r,s)+2(r-s)^j\dot\Phi(r,s)\partial_s a_{j-1}(r,s)\\ &+(r-s)^j\ddot\Phi(r,s)a_{j-1}(r,s)-(r-s)^j\partial_s^2 a_j(r,s)\end{aligned}$$

$$+ (r-s)^j \big[u_i(s) - \Phi(r,s)\big] a_j(r,s) + (r-s)^{j+1}\dot\Phi(r,s) a_j(r,s)\, ds\,. \qquad (3.2.32)$$

Since

$$\begin{aligned} I^{(j+1)}(r') := \frac{d^{j+1}}{d(r')^{j+1}} I(r') &= (-1)^{j+1}(j+1)! a_{j+1}(r,r',\mu_i,\tau_i) \\ &+ \sum_{k=0}^{j} \binom{j+1}{k} \frac{(-1)^k (j+1)!}{(j+1-k)!} (r-r')^{j+1-k} \partial_{r'}^{j+1-k} a_{j+1}(r,r',\mu_i,\tau_i) \end{aligned} \qquad (3.2.33)$$

we can see that at the point $r' = r$ one has $a_j(r,r,\mu_i,\tau_i) = cI^{(j+1)}(r)$ for some constant c. In particular, the degree of a_{j+1} is determined by the right hand side. This in turn is simple to compute since

$$\begin{aligned} I'(r') = &-(r-r')^j \dot\Phi^2(r,r') a_{j-2}(r,r') + 2(r-r')^j \dot\Phi(r,r') \partial_{r'} a_{j-1}(r,r') \\ &+ (r-r')^j \ddot\Phi(r,r') a_{j-1}(r,r') - (r-r')^j \partial_{r'}^2 a_j(r,r') \\ &+ (r-r')^j \big[u_i(r') - \Phi(r,r')\big] a_j(r,r') + (r-r')^{j+1} \dot\Phi(r,r') a_j(r,r') \end{aligned} \qquad (3.2.34)$$

(here $\dot\Phi = \partial_{r'}\Phi$). This vanishes as $r' \to r$ and so does $\frac{d^k}{d(r')^k} I(r')$ for each $1 \le k \le j$. For the $(j+1)^{th}$ derivative we find the following term-wise bounds on the degree by using the inductive hypothesis and the basic identity $\lceil x \rceil + n = \lceil x+n \rceil$ for any integer n:

$$\begin{aligned} I^{(j+1)}(r) = &-(-1)^j j! \Big(\underbrace{2\dot\Phi(r,r)\ddot\Phi(r,r) a_{j-2}(r,r)}_{\text{degree } \le \lceil 2(j+1)/3 \rceil} + \underbrace{\dot\Phi^2 \partial_{r'} a_{j-2}(r,r)}_{\text{degree } \le \lceil 2(j+1)/3 \rceil} \Big) \\ &+ 2(-1)^j j! \Big(\underbrace{\ddot\Phi(r,r) \partial_{r'} a_{j-1}(r,r)}_{\text{degree } \le \lceil (2j+1)/3 \rceil} + \underbrace{\dot\Phi(r,r) \big(\partial_{r'}^2 a_{j-1}(r,r')\big)_{r'=r}}_{\text{degree } \le \lceil 2(j+1)/3 \rceil} \Big) \\ &+ (-1)^j j! \Big(\underbrace{\dddot\Phi(r,r) a_{j-1}(r,r)}_{\text{degree } \le \lceil (2j+1)/3 \rceil} + \underbrace{\ddot\Phi(r,r) \big(\partial_{r'} a_{j-1}(r,r')\big)_{r'=r}}_{\text{degree } \le \lceil (2j+1)/3 \rceil} \Big) \\ &- (-1)^j j! \underbrace{\big(\partial_{r'}^3 a_j(r,r')\big)_{r'=r}}_{\text{degree } \le \lceil 2(j+1)/3 \rceil} \\ &+ (-1)^j j! \Big(\underbrace{\big[\partial_r u_i(r) - \dot\Phi(r,r)\big] a_j(r,r)}_{\text{degree } \le \lceil 2j/3 \rceil} + \underbrace{\big[u_i(r) - \Phi(r,r)\big] \big(\partial_{r'} a_j(r,r')\big)_{r'=r}}_{\text{degree } \le \lceil 2j/3 \rceil} \Big)\,. \end{aligned}$$

Remark 3.2.10. In addition to the inductive hypothesis and properties of the ceiling function we also use the fact that the degree of Φ is invariant under differentiation with respect to r' (c.f. (3.2.31)), whilst

$$d\big(\big[\partial_r u_i(r) - \dot{\Phi}(r,r)\big]\big) = d\big(\big[u_i(r) - \Phi(r,r)\big]\big) = 0\,. \tag{3.2.35}$$

This shows that property (1) holds in the inductive step. □

Proof of Property 2. Here we use the identity

$$\begin{aligned}(r-r')\partial_{r'}a_{j+1} = (j+1)a_{j+1} + (u_i - \Phi)a_j + (r-r')\dot{\Phi}a_j - \partial_{r'}^2 a_j + \ddot{\Phi}a_{j-1} \\ + 2\dot{\Phi}\partial_{r'}a_{j-1} - \dot{\Phi}^2 a_{j-2}\end{aligned} \tag{3.2.36}$$

which arises in the construction of the coefficients a_k (c.f. equation (3.4.31) in Section 3.4). Taking one derivative in r' and evaluating at $r' = r$ it follows that

$$\begin{aligned}&-(j+2)\big(\partial_{r'}a_{j+1}(r,r')\big)_{r'=r} = (\dot{u}_i - \dot{\Phi})a_j + (u_i - \Phi)\partial_{r'}a_j - \dot{\Phi}a_j \\ &- \partial_{r'}^3 a_j + \dddot{\Phi}a_{j-1} + 3\ddot{\Phi}\partial_{r'}a_{j-1} + 2\dot{\Phi}\partial_{r'}^2 a_{j-1} - 2\dot{\Phi}\ddot{\Phi}a_{j-2} - \dot{\Phi}^2\partial_{r'}a_{j-2}\,.\end{aligned}$$

But then Property (1) and the inductive hypothesis imply (at $r = r'$) the following degree bounds:

$$d\big((\dot{u}_i - \dot{\Phi})a_j\big) \le \lceil \frac{2j}{3} \rceil; \quad d\big((u_i - \Phi)\partial_{r'}a_j\big) \le \lceil \frac{2j+1}{3} \rceil$$

(where we use line (3.2.35) as well); further

$$d\big(\dot{\Phi}a_j\big) \le \lceil \frac{2(j+1)+1}{3} \rceil, \quad d\big(\partial_{r'}^3 a_j\big) \le \lceil \frac{2(j+1)+1}{3} \rceil$$

and

$$d\big(\dddot{\Phi}a_{j-1}\big) \le \lceil \frac{2j+1}{3} \rceil, \quad d\big(3\ddot{\Phi}\partial_{r'}a_{j-1}\big) \le \lceil \frac{2(j+1)}{3} \rceil,$$

finally for the last three terms we get

$$d\big(2\dot{\Phi}\partial_{r'}^2 a_{j-1}\big) \le \lceil \frac{2(j+1)+1}{3} \rceil, \quad d\big(2\dot{\Phi}\ddot{\Phi}a_{j-2}\big) \le \lceil \frac{2(j+1)}{3} \rceil$$

$$d\big(\dot\Phi^2\partial_{r'}a_{j-2}\big) \le \lceil \frac{2(j+1)+1}{3} \rceil\,.$$

So in summary the degree of $\big(\partial_{r'}a_{j+1}(r,r')\big)_{r'=r}$ as a polynomial in μ_i,τ_i is $\le \lceil\big(2(j+1)+1\big)/3\rceil$; this is Property (2) for the case $k=1$.

Now suppose that it holds for $1 \le i < k$ where $k \le j+1$. Starting from (3.2.36) one has

$$\begin{aligned}(\partial^k_{r'}a_{j+1})_{r'=r} &= (-\frac{1}{k}\partial^k_{r'}\big((r-r')a_{j+1}\big))_{r'=r}\\ &= -\frac{1}{k}(\partial^k_{r'}\Big((j+1)a_{j+1}+u_ia_j-\Phi a_j+(r-r')\dot\Phi a_j-\partial^2_{r'}a_j\\ &\quad +\ddot\Phi a_{j-1}+2\dot\Phi\partial_{r'}a_{j-1}-\dot\Phi^2a_{j-2}\Big))_{r'=r}\,. \end{aligned} \tag{3.2.37}$$

so that (at $r=r'$)

$$\begin{aligned}&\frac{k+j+1}{k}\partial^k_{r'}a_{j+1} =\\ &-\frac{1}{k}\partial^k_{r'}\big((u_i-\Phi)a_j+(r-r')\dot\Phi a_j-\partial^2_{r'}a_j+\ddot\Phi a_{j-1}+2\dot\Phi\partial_{r'}a_{j-1}-\dot\Phi^2a_{j-2}\big)\end{aligned} \tag{3.2.38}$$

and it remains to bound the degree of the terms on the right hand side. Always evaluating at $r=r'$ we get:

$$\partial^k_{r'}\left((u_i-\Phi)\partial_{r'}a_j\right) = \sum_{i=0}^{k}\binom{k}{i}\underbrace{\partial^i_{r'}(u_i-\Phi)\cdot\partial^{k-i+1}_{r'}a_j}_{\text{degree }\le\lceil\frac{2j+k-i+1}{3}\rceil} \tag{3.2.39}$$

(note that $d\big(\partial^i_{r'}(u_i-\Phi)\big)=0$ for each $0\le i\le k$ so that factor does not contribute to the degree of the summand). Thus the degree of the left hand side of (3.2.39) is bounded by $\lceil\,\big(2j+1+k\big)/3\,\rceil$. Also,

$$\partial^k_{r'}\big((r-r')\dot\Phi a_j\big) = -\sum_{i=0}^{k-1}\binom{k-1}{i}\underbrace{\partial^i_{r'}\dot\Phi\cdot\partial^{k-1-i}_{r'}a_j}_{\text{degree }\le\lceil\frac{2(j+1)+k-i}{3}\rceil}$$

so the left hand side's degree is bounded by $\lceil\,\big(2(j+1)+k\big)/3\,\rceil$. Likewise we can bound the degrees of the remaining terms in (3.2.38):

$$d\big(\partial^k_{r'}\left(\partial^2_{r'}a_j\right)\big) \le \lceil \frac{2(j+1)+k}{3} \rceil$$

and

$$d\big(\partial_{r'}^k(\ddot{\Phi}a_{j-1})\big) = d\Big(\sum_{i=0}^k \binom{k}{i} \underbrace{\partial_{r'}^{i+2}\Phi \cdot \partial_{r'}^{k-i}a_{j-1}}_{\text{degree } \leq \lceil \frac{2j+1+k-i}{3} \rceil}\Big) \leq \lceil \frac{2j+1+k}{3} \rceil ,$$

similarly

$$d\big(\partial_{r'}^k(2\dot{\Phi}\partial_{r'}a_{j-1})\big) = d\Big(2\sum_{i=0}^k \binom{k}{i} \underbrace{\partial_{r'}^{i+1}\Phi \cdot \partial_{r'}^{k-i+1}a_{j-1}}_{\text{degree } \leq \lceil \frac{2(j+1)+k-i}{3} \rceil}\Big) \leq \lceil \frac{2(j+1)+k}{3} \rceil$$

finally we have

$$d\big(\partial_{r'}^k(\dot{\Phi}^2 a_{j-2})\big) = d\Big(\sum_{i=0}^k \binom{k}{i} \underbrace{\partial_{r'}^{i}\dot{\Phi}^2 \cdot \partial_{r'}^{k-i}a_{j-2}}_{\text{degree } \leq \lceil \frac{2(j+1)+k-i}{3} \rceil}\Big) \leq \lceil \frac{2(j+1)+k}{3} \rceil .$$

This shows that the left hand side of (3.2.38) has degree less than $\lceil \big(2(j+1)+k\big)/3 \rceil$ and the inductive step for the first branch of statement (2) in the proposition is established.

The last part of this proof is to verify the second branch of statement (2). Here the base case is true since a_0 has degree zero; and the inductive step follows immediately from the defining equation (3.2.30) of a_{j+1} assuming that it holds for $0, 1, \ldots, j$. □

3.3 Statement and proof of the main theorem

We now prove that the formal series obtained in the previous section gives rise to a parametrix for the heat equation; this is Theorem 3.3.3 here. Essentially the outline of the argument is that of [29]; the newly arising features are due to the fact that the coefficients a_j are now polynomials in more than one variable and care has to be taken so as to maintain uniform estimates nevertheless. For each $k = 0, 1, 2, \ldots$ set

$$P_k = \phi \sum_{i=0}^{\infty} \Psi \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{(r-r')^2}{4t}\right) \exp\left(\frac{-\mu_i t}{f_1(r)f_1(r')}\right) \exp\left(\frac{-\tau_i t}{f_2(r)f_2(r')}\right) \cdot A_{i,k} \tag{3.3.1}$$

where Ψ is as defined in (3.2.15):

$$A_{i,k} = \sum_{j=0}^{k} a_j(r, r', \mu_i, \tau_i)\phi_i(x)\phi_i(y)t^{j-1/2} \tag{3.3.2}$$

is the k^{th} partial sum of the formal series given in (3.2.8) and $\phi\colon M \times M \to \mathbb{R}$ is a smooth function with compact support satisfying

$$\phi\Big(\rho\big((r,x),(r',y)\big)\Big) = \begin{cases} 1 & \text{if } d \leq R/2 \\ 0 & \text{if } d > R \end{cases}$$

where $R > 0$ and ρ denotes the distance function on M.

Lemma 3.3.1. *For each $k = 0, 1, 2, \ldots$ we have*

$$\lim_{t\to 0} P_k(t,(r,x),(\cdot,\cdot)) = \delta_{(r,x)}.$$

Proof. Let g be a smooth function on M with compact support containing the point (r,x) and contained in a coordinate neighbourhood $\big(U,(r',y)\big)$ with $U = (a,b)_{r'} \times V_y$ where V is a coordinate neighbourhood for $\mathcal{M}_1 \times \mathcal{M}_2$. Writing $d\mu$ for the Riemannian measure on M one has

$$\int_M P_k\big(t,(r,x)\big) g \, d\mu = \int_U P_k\big(t,(r,x),(r',y)\big) g(r',y)\, f_1^{d_1}(r') f_2^{d_2}(r') dy dr' .$$

Here and occasionally in the remainder of the proof the notation $P_k\big(t,(r,x)\big)$ refers to the function $(r',y) \mapsto P_k\big(t,(r,x),(r',y)\big)$. Upon substitution of (3.3.1), (3.2.15) and (3.3.2) this is equal to the long expression

$$= \int_{(a,b)} dr' \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{(r-r')^2}{4t}\Big) \times$$
$$\underbrace{\left(\sum_{j=0}^{k} \int_V \sum_{i=0}^{\infty} e^{\left(-\frac{\mu_i t}{f_1(r) f_1(r')} - \frac{\tau_i t}{f_2(r) f_2(r')}\right)} \phi_i(x)\phi_i(y) g(r',y) a_j(r,r',\mu_i,\tau_i) t^j \, dy \left(\frac{f_1(r')}{f_1(r)}\right)^{\frac{d_1}{2}} \left(\frac{f_2(r')}{f_2(r)}\right)^{\frac{d_2}{2}}\right)}_{(*)} \tag{3.3.3}$$

(we have used the compact support of g to split the integral into factors and exchange summation with integration). The presentation above shows we are applying the one - dimensional heat kernel

$$e(t,r,r') = \frac{1}{\sqrt{2\pi t}} \exp\big(-\frac{(r-r')^2}{4t}\big)$$

to the compactly supported function $(*)$. As $t \to 0$ we may therefore use the property

$$\lim_{t\to 0} e(t,r,r') = \delta_r(r') \tag{3.3.4}$$

to deduce

$$\lim_{t\to 0}\int_M P_k\big(t,(r,x)\big) g\, d\mu =$$
$$\lim_{t\to 0}\sum_{j=0}^{k}\left(\int_V \sum_{i=0}^{\infty} e^{-\frac{\mu_i t}{f_1^2(r)}} e^{-\frac{\tau_i t}{f_2^2(r)}} \phi_i(x)\phi_i(y) g(r,y) a_j(r,r,\mu_i,\tau_i)\, dy\right) t^j\,. \tag{3.3.5}$$

The coefficients of the t^j are finite (note that the a_j are polynomials in μ_i and τ_i by Lemma 3.2.8, that is finite sums, so no divergence can arise there due to the series in i). Hence, as $t \to 0$ we are left with

$$\begin{aligned}\lim_{t\to 0}\int_M P_k\big(t,(r,x)\big) g\, d\mu &= a_0 \int_V \sum_{i=0}^{\infty} \phi_i(x)\phi_i(y) g(r,y)\, dy \\ &= a_0 \sum_{i=0}^{\infty} \phi_i(x) \int_M \phi_i(y) g(r,y)\, d\mu(y) = a_0 g(r,x)\,,\end{aligned} \tag{3.3.6}$$

so the proof is complete once we normalise the coefficient a_0. □

At this point it is clear that P_k is smooth on $(0,\infty)\times\mathcal{M}_1\times\mathcal{M}_2$. The next and final Lemma shows that $(\partial_t + \triangle)P_k$ extends continuously to a function on $[0,\infty)\times\mathcal{M}_1\times\mathcal{M}_2$.

Lemma 3.3.2. *Let $T > 0$ be fixed. The following estimate holds:*

$$|(\partial_t + \triangle)P_k| \le C(f_1,f_2) t^{\alpha k - n/2 - \beta} \quad \textit{for all } t < T$$

where $\alpha, \beta > 0$ are certain constants, $n = \dim \mathcal{M}_1 + \dim \mathcal{M}_2$, and $C(f_1, f_2)$ is a smooth function that is independent of t and determined by the warping functions f_1, f_2.

Proof. We already computed $(\partial_t + \triangle)P$ (c.f. (3.2.9) and below), in particular we can see from the formulae that

$$(\partial_t + \triangle_{(r',y)})P_k = \sum_{i=0}^{\infty} \exp\big(-\frac{(r-r')^2}{4t}\big)\exp\big(\frac{-\mu_i t}{f_1(r)f_1(r')}\big)\exp\big(\frac{-\tau_i t}{f_2(r)f_2(r')}\big) \cdot \widetilde{E}_{i,k} \tag{3.3.7}$$

where initially $\widetilde{E}_{i,k}$ is a polynomial in t of degree $k + 3/2$:

$$\begin{aligned}\widetilde{E}_{i,k} = \sum_{j=0}^{k-1} \Psi\Big(&(j+1)a_{j+1} - (r-r')\partial_{r'}a_{j+1} + u_i a_j - \Phi a_j + (r-r')\dot{\Phi}a_j - \partial^2_{r'}a_j \\ &+ \ddot{\Phi}a_{j-1} + 2\dot{\Phi}\partial_{r'}a_{j-1} - \dot{\Phi}^2 a_{j-2}\Big)\phi_i(x)\phi_i(y)t^{j-1/2} \\ + \Psi\Big(&u_i a_k - \Phi a_k + (r-r')\dot{\Phi}a_k - \partial^2_{r'}a_k + \ddot{\Phi}a_{k-1} + 2\dot{\Phi}\partial_{r'}a_{k-1} - \dot{\Phi}^2 a_{k-2}\Big)\phi_i(x)\phi_i(y)t^{k-1/2} \\ + \Psi\Big(&\ddot{\Phi}a_k + 2\dot{\Phi}\partial_{r'}a_k - \dot{\Phi}^2 a_{k-1}\Big)\phi_i(x)\phi_i(y)t^{k+1/2} \\ - \Psi\Big(&\dot{\Phi}^2 a_k\Big)\phi_i(x)\phi_i(y)t^{k+3/2}.\end{aligned} \tag{3.3.8}$$

But the summands vanish identically by construction[1] *except* for those in the last three lines. Now an intermediate step in the recursive solution procedure for the a_j is the equation

$$\begin{aligned}(r-r')\partial_{r'}a_{k+1} - (k+1)a_{k+1} = u_i a_k - \Phi a_k + (r-r')\dot{\Phi}a_k - \partial^2_{r'}a_k + \ddot{\Phi}a_{k-1} \\ + 2\dot{\Phi}\partial_{r'}a_{k-1} - \dot{\Phi}^2 a_{k-2}\end{aligned} \tag{3.3.9}$$

(c.f. Section 3.4, in particular the step above is equation (3.4.31)). Substituting this into the first of the last three lines we see that we are left with estimating the expression

$$\sum_{i=0}^{\infty} \Psi \exp\big(-\frac{(r-r')^2}{4t}\big)\exp\big(\frac{-\mu_i t}{f_1(r)f_1(r')}\big)\exp\big(\frac{-\tau_i t}{f_2(r)f_2(r')}\big)\phi_i(x)\phi_i(y)\times$$

[1] that is how the a_j are determined.

$$\Big(\big((r-r')\partial_{r'} a_{k+1} - (k+1)a_{k+1} \big) t^{k-1/2} + \big(\ddot{\Phi} a_k + 2\dot{\Phi}\partial_{r'} a_k - \dot{\Phi}^2 a_{k-1} \big) t^{k+1/2} \\ - \dot{\Phi}^2 a_k t^{k+3/2} \Big) . \tag{3.3.10}$$

First, let us observe that

$$(r-r')^k \exp\big(-\frac{(r-r')^2}{4t} \big) = (2\sqrt{t})^k y^k e^{-y^2} = O\big(t^{k/2}\big) \text{ as } t \to 0_+ \tag{3.3.11}$$

with $y = \dfrac{r-r'}{2\sqrt{t}}$ (k is any non-negative integer). This expression is indeed $O\left(t^{k/2}\right)$ as $t \to 0_+$ since $y^k e^{-y^2} \to 0$ as $y \to \infty$, uniformly in (r, r'). Also, it is known that the short time asymptotic behaviour of the heat kernel on the compact factor $\mathcal{M}_1 \times \mathcal{M}_2$ is

$$\sum_i \exp\big(- (\mu_i + \tau_i)t \big) \phi_i(x)\phi_i(y) = O(t^{-n/2}) \text{ as } t \to 0_+$$

(see e.g. [36, Prop. 3.23]). Therefore

$$\sum_i \exp\big(- (\frac{\mu_i}{f_1(r)f_1(r')} + \frac{\tau_i}{f_2(r)f_2(r')})t \big) \phi_i(x)\phi_i(y) = O(t^{-n/2}) \text{ as } t \to 0_+$$

where the constant in the estimate may depend on f_1, f_2, whilst basic facts in asymptotic analysis (see for example [26, Theorem 3.2]) then imply

$$\sum_i \big(\frac{\mu_i}{f_1(r)f_1(r')} + \frac{\tau_i}{f_2(r)f_2(r')} \big)^l \exp\big(- (\frac{\mu_i}{f_1(r)f_1(r')} + \frac{\tau_i}{f_2(r)f_2(r')})t \big) \times \\ \phi_i(x)\phi_i(y) = O(t^{-n/2-l}) \text{ as } t \to 0_+ , \tag{3.3.12}$$

for any non-negative integer l, with the bound in general depending on f_1, f_2. The last estimate is important because the coefficients a_j above are polynomials in μ_i, τ_i, so in view of (3.3.12) it remains to show that their degree $d(a_j)$, can be controlled. The simple bound $d(a_j) \leq j$ is not enough to show that negative powers in t do not occur. However by Taylor expanding the coefficients and then using (3.3.11) and Lemma 3.2.8 we can see this is true. To illustrate this let us consider the contribution coming from a_{k+1} in the $t^{k-1/2}$-term in detail. Here

$$a_{k+1}(r, r', \mu_i, \tau_i) = \sum_{i=0}^{k+1} \frac{\big(\partial_{r'}^i a_{k+1}(r, r', \mu_i, \tau_i)\big)_{r'=r}}{i!} (r'-r)^i \\ + \tilde{a}_{k+1}(r, r', \mu_i, \tau_i)(r'-r)^{k+1} \tag{3.3.13}$$

with

$$\tilde{a}_{k+1}(r,r',\mu_i,\tau_i)(r'-r)^{k+1} = \frac{1}{(k+1)!}\int_r^{r'} \partial_s^{k+1} a_{k+1}(r,s,\mu_i,\tau_i)\cdot(s-r)^k\,ds$$

a polynomial in μ_i and τ_i of degree $k+1$ (c.f. Lemma 3.2.8(2)). Substituting the above into (3.3.10) gives a sum with terms of the form

$$\sum_{i=0}^{\infty} \Psi(r'-r)^i \exp\big(-\frac{(r-r')^2}{4t}\big)\times$$
$$\frac{\big(\partial^i_{r'} a_{k+1}(r,r',\mu_i,\tau_i)\big)_{r'=r}}{i!} \exp\big(-(\frac{\mu_i}{f_1(r)f_1(r')}+\frac{\tau_i}{f_2(r)f_2(r')})t\big)\phi_i(x)\phi_i(y)t^{k-1/2}$$

for $0\le i\le k$, which is $O\big(t^{i/2+k-n/2-\lceil\frac{2(k+1)+i}{3}\rceil-1/2}\big)$, and

$$\sum_{i=0}^{\infty} \Psi(r'-r)^{k+1} \exp\big(-\frac{(r-r')^2}{4t}\big)\Big(\frac{\big(\partial^{k+1}_{r'} a_{k+1}(r,r',\mu_i,\tau_i)\big)_{r'=r}}{(k+1)!}+\tilde{a}_{k+1}(r,r',\mu_i,\tau_i)\Big)\times$$
$$\exp\big(-(\frac{\mu_i t}{f_1(r)f_1(r')}+\frac{\tau_i}{f_2(r)f_2(r')})t\big)\phi_i(x)\phi_i(y)t^{k-1/2}$$

which is $O\big(t^{(k+1)/2+k-n/2-k-1-1/2}\big)=O\big(t^{k/2-n/2-1}\big)$. But note that

$$\begin{aligned}&\frac{i}{2}+k-\frac{n}{2}-\lceil\frac{2(k+1)+i}{3}\rceil-\frac{1}{2}\ge\frac{i}{2}+k-\frac{n}{2}-\frac{2(k+1)+i}{3}-1-\frac{1}{2}\\&=\frac{2k+i-4}{6}-\frac{n}{2}-\frac{3}{2}\ge\frac{k}{3}-\frac{n}{2}-\frac{13}{6}\end{aligned}$$

so overall

$$\begin{aligned}&\sum_{i=0}^{\infty}\Psi\exp\big(-\frac{(r-r')^2}{4t}\big)\exp\big(\frac{-\mu_i t}{f_1(r)f_1(r')}\big)\exp\big(\frac{-\tau_i t}{f_2(r)f_2(r')}\big)\phi_i(x)\phi_i(y)a_{k+1}t^{k-1/2}\\&=O\big(t^{\frac{k}{3}-\frac{n}{2}-\frac{13}{6}}\big)\,.\end{aligned}\tag{3.3.14}$$

Similar arguments lead to an estimate for the other terms. □

Theorem 3.3.3. *The function P_k is a parametrix for $\partial_t+\triangle$ for k large. This means that*

1. *P_k is a smooth function on $(0,\infty)\times M\times M$*

2. *$(\partial_t+\triangle)P_k$ extends to a continuous function on $[0,\infty)\times M\times M$*

3. $\lim_{t\to 0} P_k(t,(r,x),(\cdot,\cdot)) = \delta_{(r,x)}$ *is the Dirac - delta distribution based at* $(r,x) \in M$.

Proof. This has now been established in view of Lemma 3.2.7 - 3.3.2. $\square$

3.4 Computations for the parametrix

3.4.1 Applying the heat operator to the formal series

The term $\partial_t P$

$$\begin{aligned}\partial_t P = &\sum_{i=0}^{\infty} \Psi \exp\left(\frac{(r-r')^2}{kt}\right) \exp\left(\frac{-\mu_i ct}{F_1}\right) \exp\left(\frac{-\tau_i \tilde{c} t}{F_2}\right) \\ &\times \left(\partial_t A_i - \left(\frac{(r-r')^2}{kt^2} + \frac{\mu_i c}{F_1} + \frac{\tau_i \tilde{c}}{F_2}\right) A_i\right). \end{aligned} \tag{3.4.1}$$

With

$$\partial_t A_i(t,r,r',x,y) = \sum_{j=0}^{\infty} \left(j - \frac{1}{2}\right) a_j(r,r',\mu_i,\tau_i)\phi_i(x)\phi_i(y) t^{j-3/2}$$

We can collect common powers of t in the second term and write

$$\partial_t P = \sum_{i=0}^{\infty} \Psi \exp\left(\frac{(r-r')^2}{kt}\right) \exp\left(\frac{-\mu_i ct}{F_1}\right) \exp\left(\frac{-\tau_i \tilde{c} t}{F_2}\right) B_i \tag{3.4.2}$$

where

$$B_i(t,r,r',x,y) = \sum_{j=-2}^{\infty} b_j(r,r',\mu_i,\tau_i)\phi_i(x)\phi_i(y) t^{j-1/2} \tag{3.4.3}$$

with coefficients

$$b_{-2} = -\frac{(r-r')^2}{k} a_0 \tag{3.4.4}$$

$$b_{-1} = -\left(\frac{1}{2} a_0 + \frac{(r-r')^2}{k} a_1\right) \tag{3.4.5}$$

and for $j \geq 0$,

$$b_j = \left(j + \frac{1}{2}\right) a_{j+1} - \frac{(r-r')^2}{k} a_{j+2} - \left(\frac{\mu_i c}{F_1} + \frac{\tau_i \tilde{c}}{F_2}\right) a_j \,. \tag{3.4.6}$$

The term $\triangle_{(r',y)} P$

$$\triangle_{(r',y)} P =$$
$$\left[-\frac{\partial^2}{\partial(r')^2} - \left(d_1\frac{\dot f_1(r')}{f_1(r')} + d_2\frac{\dot f_2(r')}{f_2(r')}\right)\frac{\partial}{\partial r'} + \frac{1}{f_1^2(r')}\triangle_{M_1,y} + \frac{1}{f_2^2(r')}\triangle_{M_2,y})\right] P\,. \tag{3.4.7}$$

We have

$$\frac{\partial}{\partial r'}P \qquad = \sum_{i=0}^{\infty} \exp\left(\frac{(r-r')^2}{kt}\right)\exp\left(\frac{-\mu_i ct}{F_1}\right)\exp\left(\frac{-\tau_i\tilde c t}{F_2}\right)\cdot C_i \tag{3.4.8}$$

where

$$C_i(t,r,r',x,y) = \sum_{j=-1}^{\infty} c_j(r,r',\mu_i,\tau_i)\phi_i(x)\phi_i(y)t^{j-1/2} \tag{3.4.9}$$

with coefficients

$$c_{-1} = -\frac{2(r-r')}{k}\Psi a_0 \tag{3.4.10}$$
$$c_0 = -\frac{2(r-r')}{k}\Psi a_1 + a_0\partial_{r'}\Psi + \Psi\partial_{r'}a_0 \tag{3.4.11}$$

and for $j \geq 1$,

$$c_j = -\frac{2(r-r')}{k}\Psi a_{j+1} + a_j\partial_{r'}\Psi + \Psi\partial_{r'}a_j + \left(\mu_i c\frac{\partial_{r'}F_1}{F_1^2} + \tau_i\tilde c\frac{\partial_{r'}F_2}{F_2^2}\right)\Psi a_{j-1}\,. \tag{3.4.12}$$

From this we get

$$\begin{aligned}
&\frac{\partial^2}{\partial(r')^2}P \\
&= \sum_{i=0}^{\infty}\left(-\frac{2(r-r')}{kt} + \mu_i ct\frac{\partial_{r'}F_1}{F_1^2} + \tau_i\tilde c t\frac{\partial_{r'}F_2}{F_2^2}\right)\exp\left(\frac{(r-r')^2}{kt}\right)\exp\left(\frac{-\mu_i ct}{F_1}\right)\exp\left(\frac{-\tau_i\tilde c t}{F_2}\right)\cdot C_i \\
&+ \sum_{i=0}^{\infty}\exp\left(\frac{(r-r')^2}{kt}\right)\exp\left(\frac{-\mu_i ct}{F_1}\right)\exp\left(\frac{-\tau_i\tilde c t}{F_2}\right)\cdot\partial_{r'}C_i \\
&= \sum_{i=0}^{\infty}\exp\left(\frac{(r-r')^2}{kt}\right)\exp\left(\frac{-\mu_i ct}{F_1}\right)\exp\left(\frac{-\tau_i\tilde c t}{F_2}\right)\cdot D_i
\end{aligned} \tag{3.4.13}$$

where

$$D_i(t,r,r',x,y) = \sum_{j=-2}^{\infty} \delta_j(r,r',\mu_i,\tau_i)\phi_i(x)\phi_i(y)t^{j-1/2} \tag{3.4.14}$$

with coefficients

$$\delta_{-2} = -\frac{2(r-r')}{k}c_{-1} \tag{3.4.15}$$

$$\delta_{-1} = -\frac{2(r-r')}{k}c_0 + \partial_{r'}c_{-1} \tag{3.4.16}$$

and for $j \geq 0$,

$$\delta_j = -\frac{2(r-r')}{k}c_{j+1} + \partial_{r'}c_j + \left(\mu_i c\frac{\partial_{r'}F_1}{F_1^2} + \tau_i\tilde{c}\frac{\partial_{r'}F_2}{F_2^2}\right)c_{j-1}\,. \tag{3.4.17}$$

Finally,

$$\triangle_{M_1,y} P = \sum_{i=0}^{\infty} \mu_i \Psi \exp\left(\frac{(r-r')^2}{kt}\right)\exp\left(\frac{-\mu_i ct}{F_1}\right)\exp\left(\frac{-\tau_i\tilde{c}t}{F_2}\right)\cdot A_i \tag{3.4.18}$$

and

$$\triangle_{M_2,y} P = \sum_{i=0}^{\infty} \tau_i \Psi \exp\left(\frac{(r-r')^2}{kt}\right)\exp\left(\frac{-\mu_i ct}{F_1}\right)\exp\left(\frac{-\tau_i\tilde{c}t}{F_2}\right)\cdot A_i\,. \tag{3.4.19}$$

Collecting the terms from (3.4.3), (3.4.8), (3.4.13), (3.4.18) and (3.4.19) we get

$$(\partial_t + \triangle_{(r',y)})P = \sum_{i=0}^{\infty} \exp\left(\frac{(r-r')^2}{kt}\right)\exp\left(\frac{-\mu_i ct}{F_1}\right)\exp\left(\frac{-\tau_i\tilde{c}t}{F_2}\right)\cdot E_i \tag{3.4.20}$$

where

$$E_i(t,r,r',x,y) = \sum_{j=-2}^{\infty} e_j(r,r',\mu_i,\tau_i)\phi_i(x)\phi_i(y)t^{j-1/2} \tag{3.4.21}$$

with coefficients

$$e_{-2} = \Psi b_{-2} - \delta_{-2} \tag{3.4.22}$$

$$e_{-1} = \Psi b_{-1} - \delta_{-1} - (d_1\frac{\dot{f}_1}{f_1} + d_2\frac{\dot{f}_2}{f_2})c_{-1} \tag{3.4.23}$$

and for $j \geq 0$,

$$e_j = \Psi\left(b_j + \left(\frac{\mu_i}{f_1^2} + \frac{\tau_i}{f_2^2}\right)a_j\right) - \delta_j - \left(d_1\frac{\dot{f}_1}{f_1} + d_2\frac{\dot{f}_2}{f_2}\right)c_j\,. \tag{3.4.24}$$

3.4.2 Solving for the coefficients

$\mathbf{e_{-2} = 0}$

$$
\begin{aligned}
e_{-2} &= \Psi b_{-2} - \delta_{-2} = -\Psi \frac{(r-r')^2}{k} a_0 + \frac{2(r-r')}{k} c_{-1} \\
&= -\Psi \frac{(r-r')^2}{k} a_0 - \frac{4(r-r')^2}{k^2} \Psi a_0 = -\Psi a_0 \frac{(r-r')^2}{k} \left(1 + \frac{4}{k}\right) = 0\,.
\end{aligned}
$$

Thus we see that $k = -4$.

$\mathbf{e_{-1} = 0}$

Setting $\Psi(r,r') = \Big(f_1(r) f_1(r')\Big)^{-d_1/2} \Big(f_2(r) f_2(r')\Big)^{-d_2/2}$ and $a_0 = \text{const}$ gives

$$
\begin{aligned}
e_{-1} &= \Psi b_{-1} - \delta_{-1} - \left(d_1 \frac{\dot f_1}{f_1} + d_2 \frac{\dot f_2}{f_2}\right) c_{-1} \\
&= \Psi \left(\frac{(r-r')^2}{4} a_1 - \frac{1}{2} a_0\right) - \left(\frac{r-r'}{2} c_0 + \partial_{r'} c_{-1}\right) - \left(d_1 \frac{\dot f_1}{f_1} + d_2 \frac{\dot f_2}{f_2}\right) c_{-1} \\
&= \left(\frac{(r-r')^2}{4} \Psi a_1 - \frac{1}{2} \Psi a_0\right) - \frac{r-r'}{2} \left(\frac{r-r'}{2} \Psi a_1 + a_0 \partial_{r'} \Psi + \Psi \partial_{r'} a_0\right) \\
&- \partial_{r'} \left(\frac{r-r'}{2} \Psi a_0\right) - \left(d_1 \frac{\dot f_1}{f_1} + d_2 \frac{\dot f_2}{f_2}\right) \left(\frac{r-r'}{2} \Psi a_0\right) \\
&= -(r-r') a_0 \left(\partial_{r'} \Psi + \left(\frac{d_1}{2} \frac{\dot f_1}{f_1} + \frac{d_2}{2} \frac{\dot f_2}{f_2}\right) \Psi\right) - (r-r') \Psi \partial_{r'} a_0 = 0\,.
\end{aligned}
$$

For later purposes we note that

$$
\dot\Psi = \partial_{r'} \Psi = -\left(\frac{d_1}{2} \frac{\dot f_1(r')}{f_1(r')} + \frac{d_2}{2} \frac{\dot f_2(r')}{f_2(r')}\right) \Psi
$$

$\mathbf{e_j = 0}$ for $\mathbf{j \geq 0}$

We set $F_1(r,r') = f_1(r)f_1(r')$ and $F_2(r,r') = f_2(r)f_2(r')$, and $c = \tilde{c} = 1$.

$$
\begin{aligned}
e_j &= \Psi\left(b_j + \left(\frac{\mu_i}{f_1^2} + \frac{\tau_i}{f_2^2}\right)a_j\right) - \delta_j - \left(d_1\frac{\dot{f}_1}{f_1} + d_2\frac{\dot{f}_2}{f_2}\right)c_j \\
&= \Psi\left[\left(j+\frac{1}{2}\right)a_{j+1} + \frac{(r-r')^2}{4}a_{j+2} - \left(\frac{\mu_i}{f_1(r)f_1(r')} + \frac{\tau_i}{f_2(r)f_2(r')}\right)a_j \right.\\
&+ \left.\left(\frac{\mu_i}{f_1(r')^2} + \frac{\tau_i}{f_2^2}\right)a_j\right] - \left[\frac{(r-r')}{2}c_{j+1} + \partial_{r'}c_j \right.\\
&+ \left.\left(\mu_i\frac{\dot{f}_1(r')}{f_1(r)f_1^2(r')} + \tau_i\frac{\dot{f}_2(r')}{f_2(r)f_2^2(r')}\right)c_{j-1}\right] - \left(d_1\frac{\dot{f}_1}{f_1} + d_2\frac{\dot{f}_2}{f_2}\right)c_j\,.
\end{aligned}
$$

Write

$$\Phi = \frac{\mu_i}{f_1(r)f_1(r')} + \frac{\tau_i}{f_2(r)f_2(r')} \tag{3.4.25}$$

so that

$$\dot{\Phi} = \partial_{r'}\Phi = -\Big(\mu_i\frac{\dot{f}_1(r')}{f_1(r)f_1^2(r')} + \tau_i\frac{\dot{f}_2(r')}{f_2(r)f_2^2(r')}\Big)\,.$$

Proceeding with the above and simplifying,

$$
\begin{aligned}
&= \Psi\left((j+1)\,a_{j+1} - \Phi a_j + \left(\frac{\mu_i}{f_1(r')^2} + \frac{\tau_i}{f_2(r')^2}\right)a_j\right) \\
&+ \Psi(r-r')\left(\dot{\Phi}a_j - \partial_{r'}a_{j+1}\right) \\
&+ \Psi\left(\left(\frac{d_1}{2}\frac{\dot{f}_1(r')}{f_1(r')} + \frac{d_2}{2}\frac{\dot{f}_2(r')}{f_2(r')}\right)^2 a_j + a_j\partial_{r'}\left(\frac{d_1}{2}\frac{\dot{f}_1(r')}{f_1(r')} + \frac{d_2}{2}\frac{\dot{f}_2(r')}{f_2(r')}\right)\right. \\
&- \left.\partial_{r'}^2 a_j + \ddot{\Phi}a_{j-1}\right) + \Psi\left(2\dot{\Phi}\partial_{r'}a_{j-1} - \dot{\Phi}^2 a_{j-2}\right)\,.
\end{aligned}
$$

To further clarify notation, write

$$\Theta = \left(\frac{d_1}{2}\frac{\dot{f}_1(r')}{f_1(r')} + \frac{d_2}{2}\frac{\dot{f}_2(r')}{f_2(r')}\right) \tag{3.4.26}$$

in the above and re - order the terms,

$$
\begin{aligned}
e_j = \Psi\Big(&(j+1)a_{j+1} - (r-r')\partial_{r'}a_{j+1} + u_i a_j - \Phi a_j + (r-r')\dot{\Phi}a_j \\
&- \partial_{r'}^2 a_j + \ddot{\Phi}a_{j-1} + 2\dot{\Phi}\partial_{r'}a_{j-1} - \dot{\Phi}^2 a_{j-2}\Big)
\end{aligned} \tag{3.4.27}
$$

where $a_{-2} = a_{-1} = 0$ and

$$u_i = u_i(r', \mu_i, \tau_i) = \Theta^2 + \dot{\Theta} + \left(\frac{\mu_i}{f_1(r')^2} + \frac{\tau_i}{f_2(r')^2} \right) . \tag{3.4.28}$$

As in [29], we set (3.4.27) equal to zero and solve for the a_j, $j \geq 1$ successively.

$$\begin{aligned}
&0 = a_1 - (r - r')\partial_{r'} a_1 + u_i a_0 - \Phi a_0 + (r - r')\dot{\Phi} a_0 \\
&\implies \\
&\frac{\partial}{\partial r'} \left[(r - r') a_1 \right] = u_i a_0 - \Phi a_0 + (r - r')\dot{\Phi} a_0 \\
&\implies \\
&a_1(r, r', \mu_i, \tau_i) = \frac{a_0}{r - r'} \int_r^{r'} u_i(s, \mu_i, \tau_i) - \Phi(r, s, \mu_i, \tau_i) + (r - s)\dot{\Phi}(r, s, \mu_i, \tau_i)\, ds \\
&= \frac{a_0}{r - r'} \int_r^{r'} u_i(s, \mu_i, \tau_i)\, ds - \frac{a_0}{r - r'} \int_r^{r'} \Phi(r, s, \mu_i, \tau_i)\, ds \\
&\quad + \frac{a_0}{r - r'} \int_r^{r'} (r - s)\dot{\Phi}(r, s, \mu_i, \tau_i)\, ds \\
&\overset{ibp}{=} \frac{a_0}{r - r'} \int_r^{r'} u_i(s, \mu_i, \tau_i)\, ds - \frac{a_0}{r - r'} \int_r^{r'} \Phi(r, s, \mu_i, \tau_i)\, ds \\
&\quad + \frac{a_0}{r - r'} \left((r - r')\Phi + \int_r^{r'} \Phi\, ds \right) \\
&= \frac{a_0}{r - r'} \int_r^{r'} u_i(s, \mu_i, \tau_i)\, ds + a_0 \Phi\, .
\end{aligned} \tag{3.4.29, 3.4.30}$$

Similarly we determine

$$\begin{aligned}
\frac{\partial}{\partial r'} \left[(r - r')^2 a_2 \right] &= (r - r') \left(u_i - \Phi + (r - r')\dot{\Phi} \right) a_1 - (r - r')\partial_{r'}^2 a_1 \\
&\quad + \ddot{\Phi}(r - r') a_0
\end{aligned}$$

which, upon substitution of (3.4.30) for a_1 becomes

$$\begin{aligned}
&= a_0 \left(u_i - \Phi + (r - r')\dot{\Phi} \right) \left(\int_r^{r'} u_i(s, \mu_i, \tau_i)\, ds + (r - r')\Phi \right) \\
&\quad - (r - r')\partial_{r'}^2 a_1 + \ddot{\Phi}(r - r') a_0\, .
\end{aligned}$$

then substituting the derivative $\partial^2_{r'} a_1$, using again (3.4.30), gives

$$
\begin{aligned}
&= a_0 u_i \int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds + a_0(r-r')u_i\Phi - a_0\Phi \int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds \\
&- a_0(r-r')\Phi^2 + a_0(r-r')\dot{\Phi}\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds + a_0(r-r')^2\Phi\dot{\Phi} \\
&- (r-r')\partial^2_{r'} a_1 + \ddot{\Phi}(r-r')a_0\,.
\end{aligned}
$$

Finally we substitute for the starred expression below,

$$
\begin{aligned}
&= \frac{a_0}{2}\partial_{r'}\left(\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds\right)^2 + a_0\partial_{r'}\left((r-r')\Phi\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds\right) \\
&+ \frac{a_0}{2}\partial_{r'}\big((r-r')\Phi\big)^2 \underbrace{-(r-r')\partial^2_{r'}a_1 + a_0(r-r')\ddot{\Phi}}_{(*)}
\end{aligned}
$$

$$
\left((*) = a_0\partial_{r'}u_i + \frac{2a_0}{r-r'}u_i + \frac{2a_0}{(r-r')^2}\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds + a_0(r-r')\ddot{\Phi}\right)
$$

to get

$$
\begin{aligned}
&= \frac{a_0}{2}\partial_{r'}\left(\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds\right)^2 + a_0\partial_{r'}\left((r-r')\Phi\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds\right) \\
&+ \frac{a_0}{2}\partial_{r'}\big((r-r')\Phi\big)^2 - \left(a_0\partial_{r'}u_i + \frac{2a_0}{r-r'}u_i + \frac{2a_0}{(r-r')^2}\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds\right)\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(r-r')^2 a_2 = \\
&\int_r^{r'} (r-s)\Big(u_i(s,\mu_i,\tau_i) - \Phi(r,s,\mu_i,\tau_i) + (r-s)\dot{\Phi}(r,s,\mu_i,\tau_i)\Big)\, a_1(r,s,\mu_i,\tau_i)\,ds \\
&\qquad - \int_r^{r'} (r-s)\partial^2_s a_1(r,s,\mu_i,\tau_i)\,ds + a_0\int_r^{r'} \ddot{\Phi}(r,s,\mu_i\tau_i)(r-s)\,ds \\
&\implies \\
&(r-r')^2 a_2(r,r',\mu_i,\tau_i) =
\end{aligned}
$$

$$\frac{a_0}{2}\left(\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds\right)^2 + a_0(r-r')\Phi\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds$$
$$+\frac{a_0}{2}(r-r')^2\Phi^2 - \int_r^{r'}(r-s)\partial_s^2 a_1(r,s,\mu_i,\tau_i)\,ds$$
$$+a_0\int_r^{r'}(r-s)\ddot{\Phi}(r,s,\mu_i,\tau_i)\,ds$$
$$=\frac{a_0}{2}\left(\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds\right)^2 + a_0(r-r')\Phi\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds$$
$$+\frac{a_0}{2}(r-r')^2\Phi^2 - \int_r^{r'}(r-s)\partial_s^2 a_1(r,s,\mu_i,\tau_i)\,ds$$
$$+a_0\int_r^{r'}(r-s)\ddot{\Phi}(r,s,\mu_i,\tau_i)\,ds$$
$$\overset{ibp}{=}\frac{a_0}{2}\left(\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds\right)^2 + a_0(r-r')\Phi\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds$$
$$+\frac{a_0}{2}(r-r')^2\Phi^2 - [(r-r')\partial_{r'}a_1 + a_1(r,r',\mu_i,\tau_i) - a_1(r,r,\mu_i,\tau_i)]$$
$$+a_0\Big[(r-r')\dot{\Phi} + \Phi(r,r',\mu_i,\tau_i) - \Phi(r,r,\mu_i,\tau_i)\Big]$$
$$\Longrightarrow$$
$$a_2 = \frac{a_0}{2(r-r')^2}\left(\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds\right)^2 + \frac{a_0}{(r-r')}\Phi\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds$$
$$-\frac{2a_0}{(r-r')^3}\int_r^{r'} u_i(s,\mu_i,\tau_i)\,ds - \frac{a_0}{(r-r')^2}\big[u_i(s,\mu_i,\tau_i) + u_i(r',\mu_i,\tau_i)\big] + \frac{a_0}{2}\Phi^2$$

and in general,

$$(r-r')\partial_{r'}a_{j+1} - (j+1)a_{j+1} =$$
$$u_i a_j - \Phi a_j + (r-r')\dot{\Phi}a_j - \partial_{r'}^2 a_j + \ddot{\Phi}a_{j-1} + 2\dot{\Phi}\partial_{r'}a_{j-1} - \dot{\Phi}^2 a_{j-2} \tag{3.4.31}$$
$$\Longrightarrow$$
$$\frac{\partial}{\partial r'}\left[(r-r')^{j+1}a_{j+1}\right] =$$
$$-(r-r')^j\dot{\Phi}^2 a_{j-2} + 2(r-r')^j\dot{\Phi}\partial_{r'}a_{j-1} + (r-r')^j\ddot{\Phi}a_{j-1}$$

$$- (r-r')^j \partial_{r'}^2 a_j + (r-r')^j \big(u_i - \Phi\big) a_j + (r-r')^{j+1} \dot{\Phi} a_j \tag{3.4.32}$$

$$\implies$$

$$a_{j+1}(r, r', \mu_i, \tau_i) =$$

$$\frac{1}{(r-r')^{j+1}} \int_r^{r'} \Bigg(- (r-s)^j \dot{\Phi}^2(r,s) a_{j-2}(r,s) + 2(r-s)^j \dot{\Phi}(r,s) \partial_s a_{j-1}(r,s)$$

$$+ (r-s)^j \ddot{\Phi}(r,s) a_{j-1}(r,s) - (r-s)^j \partial_s^2 a_j(r,s)$$

$$+ (r-s)^j \big[u_i(s) - \Phi(r,s)\big] a_j(r,s) + (r-s)^{j+1} \dot{\Phi}(r,s) a_j(r,s) \Bigg) ds \tag{3.4.33}$$

(The dependence of the terms on the eigenvalues μ_i, τ_i has been suppressed since it is not relevant for the integration.)

Chapter 4

Explicit formulae for resolvent symbols and their application

4.1 Introduction

Let (M, g) be a closed Riemannian manifold of dimension n and denote by $\triangle$ the corresponding Laplace Beltrami operator. It is well - known (see for example [36, Prop. 3.23]) that there exists a short - time asymptotic expansion of the heat kernel $k_\triangle(t, x, y)$ on M along the diagonal,

$$\operatorname{tr}(k_\triangle(t, x, x)) \backsim_{t \to 0_+} \sum_{j \geq 0} c_j(x) t^{\frac{j-n}{2}} . \tag{4.1.1}$$

The heat kernel was described in Chapter 3 as the fundamental solution to the heat equation $\partial_t + \triangle_y$ on M; but here we shall use the equivalent formulation of $k_\triangle(t, x, y)$ as the Schwartz kernel of the heat operator $e^{-t\triangle}$ (the Schwartz kernel of a pseudodifferential operator T refers to the family of distributions $k_T(x, \cdot)$, parametrised by $x \in M$, that satisfies the identity $Tf(x) = \langle k(x, \cdot), f \rangle$ for $f \in C^\infty(M)$ where $f \mapsto \langle u, f \rangle$ denotes the application of the distribution u to the function f). The heat operator $e^{-t\triangle}$ is defined as a Cauchy integral via the holomorphic functional calculus by

$$e^{-t\triangle} := \frac{i}{2\pi}\int_\gamma e^{-t\lambda}(\triangle - \lambda)^{-1}\,d\lambda \tag{4.1.2}$$

where the contour γ consists of two rays $\{\, re^{\pm i\sigma} \mid r \geq \delta \,\}$ in the first and fourth quadrant of the complex plane ($\delta > 0$) respectively, which are connected via an open arc $\{\delta e^{i\theta} \mid -\sigma \leq \theta \leq \sigma\}$ that encircles the origin. In particular, γ properly encloses the positive real axis which contains the spectrum of $\triangle$. Integrating the expansion (4.1.1) over the manifold gives rise to the short time asymptotic expansion of the heat trace

$$\mathrm{Tr}\left(e^{-t\triangle}\right) \quad \backsim \quad \sum_{j\geq 0} c_j\, t^{\frac{j-n}{2}} \qquad t \to 0_+ \tag{4.1.3}$$

where the coefficients $c_j = \int_M c_j(x)\,dx$ yield geometric information about the underlying manifold. For the coefficients with even index the formulas

$$c_{2k} = \int_M \mathrm{tr}\left(c_{2k}(x)\right)\,dx\,, \tag{4.1.4}$$

where

$$c_{2k}(x) = \int_{\mathbb{R}^n}\int_\gamma e^{-\lambda}\, r_{-2-2k}(x,\xi,\lambda)\,đ\lambda đ\xi \tag{4.1.5}$$

are well known, here $đ\lambda = id\lambda/2\pi$ and $đ\xi = d\xi/(2\pi)^n$ denotes rescaled Lebesgue measure. Furthermore one can show that the coefficients with odd indices vanish (c.f. [38], [17], see also Section 4.2.3 below for the first odd coefficient). Let us explain in a little more detail the term $r_{-2-2k}(x,\xi,\lambda)$ in the integrand as it is important in the sequel. These are called the *resolvent symbols* of the operator under consideration (in our case $\triangle$); they arise in the asymptotic expansion of the local symbol of the resolvent operator $(\triangle - \lambda)^{-1}$,

$$r(x,\xi,\lambda) \quad \backsim \quad \sum_{j\geq 0} r_{-2-j}(x,\xi,\lambda)\,, \tag{4.1.6}$$

which is valid for $|\xi| + |\lambda|^{1/2} \geq 1$ and λ in a suitable sector $\Lambda \subset \mathbb{C}$. Here, the left hand side is an element of the class CS^{-2} of parameter dependent classical symbols as defined in [41, §9], while each summand r_{-2-j} on the right hand side belongs to CS^{-2-j}. The relation $\backsim$ means that for each $N = 1, 2, \ldots$ the difference

$$r(x, \xi, \lambda) \quad - \quad \sum_{j=0}^{N-1} r_{-2-j}(x, \xi, \lambda)$$

lies in the symbol class CS^{-2-N} (Symbols classes are introduced in more detail in Chapter 5). The asymptotic expansion (4.1.6) arises in the construction of the parametrix for $(\triangle - \lambda)^{-1}$ where one obtains the following well - known recursive formulae of the resolvent symbols:

$$\begin{aligned} r_{-2} &= (a_2 - \lambda)^{-1}\,, \\ r_{-2-j} &= -r_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!} (\partial_\xi^\mu a_{2-k})(D_x^\mu r_{-2-l}) \qquad (j \geq 1) \end{aligned}$$

here the functions a_2, a_1, a_0 constitute the homogeneous summands of the symbol $\sigma_\triangle$ of the operator $\triangle$, that is

$$\sigma_\triangle(x, \xi) = a_2(x, \xi) + a_1(x, \xi) + a_0(x, \xi) \quad \text{with} \quad a_k(x, \alpha\xi) = \alpha^k a_k(x, \xi)\,,$$

furthermore μ denotes a multi-index and $D_{x_i} = -i\partial/\partial x_i$. In particular we see that the functions r_{-2-j} are determined by the local symbol of the Laplacian and a finite number of their derivatives. In Section (4.2.2) we shall provide explicit formulae for the first terms in the asymptotic expansion (c.f.Theorem 4.2.1), to the best of our knowledge these do not appear elsewhere in the literature. The reason we are interested in these closed formulas is that they facilitate via (4.1.5) a direct and elementary calculation of the heat coefficients in the asymptotic expansion of the heat trace; this is illustrated in Section 4.2.3 where we apply our result to recover well - known geometric expressions for the first three heat coefficients.

A further application of resolvent symbols is that they are effective for deriving index formulae. Let M be a smooth compact manifold without boundary of even dimension $n = 2k$ with vector bundles $\mathcal{E}^\pm \xrightarrow{\pi} M$ and consider a first - order elliptic differential operator

$$D\colon C^\infty(M,\mathcal{E}^+)\to C^\infty(M,\mathcal{E}^-) \tag{4.1.7}$$

acting on smooth sections, with corresponding Laplacians

$$\triangle = D^*D\colon C^\infty(M,\mathcal{E}^+)\to C^\infty(M,\mathcal{E}^+) \tag{4.1.8}$$

$$\widetilde{\triangle} = DD^*\colon C^\infty(M,\mathcal{E}^-)\to C^\infty(M,\mathcal{E}^-)\,. \tag{4.1.9}$$

It was observed by H.P. McKean and I.M. Singer [24] that the index of D, defined as

$$\operatorname{ind} D := \dim\ker D - \dim\operatorname{coker} D$$

satisfies the identity

$$\begin{aligned}\operatorname{ind} D &= \operatorname{Tr}(e^{-t\triangle}) - \operatorname{Tr}(e^{-t\widetilde{\triangle}})\\ &= \int_M \operatorname{tr}\Big(k_\triangle(t,x,x)\Big) - \operatorname{tr}\Big(k_{\widetilde{\triangle}}(t,x,x)\Big)\ |dx|\end{aligned} \tag{4.1.10}$$

where $k_\triangle(t,x,x)$ is the Schwartz kernel of the heat operator $e^{-t\triangle}$ described above (and likewise for $\widetilde{\triangle}$). If M is a Riemannian spin manifold and D is of Dirac - type, that is

$$D = \not{D}\otimes I + I\otimes\nabla^{\mathcal{F}}\colon\quad C^\infty(M,\mathcal{S}^+\otimes\mathcal{F})\longrightarrow C^\infty(M,\mathcal{S}^-\otimes\mathcal{F})$$

where $\mathcal{S}^\pm$ denotes the spinor bundle and $\mathcal{F}\to M$ is some coefficient bundle with connection $\nabla^{\mathcal{F}}$, then the Atiyah-Singer index theorem states that

$$\operatorname{ind} D = \frac{1}{(2\pi)^{n/2}}\int_M \widehat{A}(M)\operatorname{ch}(\mathcal{F}) \tag{4.1.11}$$

where

$$\widehat{A}(M) = \det{}^{1/2}\frac{R/2}{\sinh R/2} \tag{4.1.12}$$

is the $\widehat{A}$ - genus form with respect to Riemannian curvature R whilst

$$\operatorname{ch}(\mathcal{F}) = \operatorname{tr} e^{-(\nabla^{\mathcal{F}})^2} \tag{4.1.13}$$

denotes the Chern character of the coefficient bundle $\mathcal{F}$ (here $(\nabla^{\mathcal{F}})^2$ is the curvature of the connection $\nabla^{\mathcal{F}}$).
One approach to establish the equality in (4.1.11) is via the McKean Singer formula (4.1.10). Here one has to show that, pointwise, the limit

$$\lim_{t\to 0_+}\Big(\mathrm{tr}(k_{\triangle}(t,x,x)) - \mathrm{tr}(k_{\widetilde{\triangle}}(t,x,x))\Big) \tag{4.1.14}$$

is finite and equal to the index density. As a starting point one establishes asymptotic expansions

$$\mathrm{tr}(k_{\triangle}(t,x,x)) \backsim_{t\to 0_+} \sum_{j\geq 0} c_j(x) t^{\frac{j-n}{2}} \quad \text{and} \quad \mathrm{tr}(k_{\widetilde{\triangle}}(t,x,x)) \backsim_{t\to 0_+} \sum_{j\geq 0} \widetilde{c}_j(x) t^{\frac{j-n}{2}}, \tag{4.1.15}$$

then to get to the existence of the limit (4.1.14) one has to establish that

$$c_j(x) - \widetilde{c}_j(x) = 0 \qquad \text{for } j < n \tag{4.1.16}$$

so that the negative powers in t vanish, allowing the limit (4.1.14) to exist, which one then has to compute.

One way to deduce that the required limit is finite is to estimate the heat kernel by an application of the "Duhamel principle" [24]. This method requires knowledge of the full heat kernel which is a global object (i.e. well defined on the whole manifold). On the other hand the required limit (4.1.14) is local since one is concerned with short time evolution as $t \to 0_+$, so one might wonder whether knowledge of the complete heat kernel is actually necessary to get to the finiteness of the limit. Another approach was proposed by E. Getzler [13]. It starts with the observation that the heat kernel of the harmonic oscillator $D^2 = -\frac{d^2}{dx^2} + a^2x^2$ coincides with $\widehat{A}(x)$. Then, by scaling the variables, the heat trace associated with $\triangle$ is reduced to the heat trace associated with D^2 and via this identification one proceeds to compute $\lim_{t\to 0_+}\Big(\mathrm{tr}(k_{\triangle}(t,x,x)) - \mathrm{tr}(k_{\widetilde{\triangle}}(t,x,x))\Big) = \widehat{A}(x)$.
In the second part of this chapter we shall study an alternative approach which in some sense is simpler and more direct. The idea is to establish a correspondence

between the generating function for the characteristic numbers and a generating function built out of the terms in the asymptotic expansion of the local resolvent symbols. Let us introduce the generating function for the characteristic numbers by way of an example. Suppose M is a Kähler manifold of complex dimension n and let $\mathcal{W} \to M$ be a holomorphic vector bundle of rank N. In this case the index form is identified with a Todd class form and the Atiyah - Singer index theorem specialises to the Hirzebruch - Riemann - Roch theorem

$$\chi(M, \mathcal{W}) = \frac{1}{(2\pi i)^{n/2}} \int_M \mathrm{Td}(M)\mathrm{ch}(\mathcal{W})\,. \tag{4.1.17}$$

Here

$$\chi(M, \mathcal{W}) := \sum_{j=0}^{n} (-1)^j \dim H^j(M, \Omega(\mathcal{W})) \tag{4.1.18}$$

is the Euler characteristic, $H^*(M, \Omega(\mathcal{W}))$ denotes Dolbeaut cohomology. On the right hand side,

$$\mathrm{Td}(M) = \mathrm{Td}_1 + \cdots + \mathrm{Td}_n \tag{4.1.19}$$

is the Todd class defined by the Todd polynomials Td_i. These are polynomials in the Chern classes $\mathrm{c}_1, \ldots, \mathrm{c}_n$ of M and obtained from the generating function

$$\mathrm{Td}(M, tR) := \det\left(\frac{tR}{e^{tR} - 1}\right) = 1 + \mathrm{Td}_1(R)\,t + \mathrm{Td}_2(R)\,\frac{t^2}{2!} + \ldots \tag{4.1.20}$$

where R is the curvature of a hermitian connection on the tangent bundle TM, whilst a representative of each Chern class $c_k(R)$ is given as the coefficient of the generating function

$$\det(1 + tR) = 1 + c_1(R)t + c_2(R)\frac{t^2}{2!} + \ldots \tag{4.1.21}$$

Lastly, the Chern character of $\mathcal{W}$ with connection $\nabla^{\mathcal{W}}$ is the series

$$\mathrm{ch}(\mathcal{W}) = \mathrm{tr}\, e^{-(\nabla^{\mathcal{W}})^2} = \sum_{k=0}^{\infty} \frac{\mathrm{tr}\left((\nabla^{\mathcal{W}})^{2k}\right)}{k!}\,. \tag{4.1.22}$$

The generating function on the analytical side can be represented in three essentially equivalent ways (for simplicity we state the results in terms of the operator $\triangle$, but the statements hold of course true in like manner for $\widetilde{\triangle}$). First, there is the short time asymptotic expansion of the heat trace already mentioned in (4.1.15),

$$\operatorname{Tr}\left(e^{-t\triangle}\right) \quad \backsim_{t\to 0_+} \quad \sum_{k\geq 0} c_k\, t^{\frac{k-n}{2}}\,. \tag{4.1.23}$$

Secondly one may consider the resolvent $(\triangle-\lambda)^{-1}$, whose N^{th} power is trace class whenever $N > n/2$. If we restrict λ to suitable rays then there exists an asymptotic expansion [16]

$$\operatorname{Tr}\left((\triangle-\lambda)^{-N}\right) \quad \backsim \quad \sum_{k\geq 0} c_k^N (-\lambda)^{\frac{n-k}{2}-N} \qquad \lambda\to\infty\,. \tag{4.1.24}$$

We note that for any N,

$$\int_\gamma e^{-t\lambda}(\triangle-\lambda)^{-1}\, d\lambda \;=\; (N+1)!\int_\gamma e^{-t\lambda}(\triangle-\lambda)^{-N}\, d\lambda\,. \tag{4.1.25}$$

This relates the coefficients c_k in (4.1.23) to the coefficients c_k^N in (4.1.24) (in fact, one can show that they differ by a constant). Finally, one may consider the trace of the power operator

$$\triangle^{-s} = \frac{i}{2\pi}\int_\gamma \lambda^{-s}(\triangle-\lambda)^{-1}d\lambda \tag{4.1.26}$$

This has a classical trace $\operatorname{Tr}(\triangle^{-s})$ for $\operatorname{Re}(s)$ large enough. If we denote by $\zeta(\triangle,s)$ its meromorphic extension to $\mathbb{C}$ then the pole structure is commonly represented by the relation

$$\Gamma(s)\,\zeta(\triangle,s) \quad \backsim \quad \sum_{k\geq 0}\frac{c_k}{s+\frac{k-n}{2}} - \frac{\dim\operatorname{Ker}\triangle}{s} \tag{4.1.27}$$

where $\backsim$ means that the left hand side is a meromorphic function on $\mathbb{C}$ whose poles are indicated in the right hand side. Furthermore, the Gamma function $\Gamma(s)$ is the meromorphic extension of the integral $\int_0^\infty t^{s-1}e^{-t}\,dt$ (initially defined for Re(s)>0) to all of $\mathbb{C}$, and $\operatorname{Ker}\triangle$ denotes the Nullspace of $\triangle$. The Mellin transform relates (4.1.23) to (4.1.27),

$$\zeta(\triangle,s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{1-s}\operatorname{Tr}(e^{-t\triangle})\,ds \tag{4.1.28}$$

and the coefficients c_k in these expansions coincide (as suggested by the notation). A precise account of the equivalence between these generating functions is given for example in [38], [17]. Here we will be focusing on the asymptotic expansion of the heat trace and the formulae (4.1.4) - (4.1.5) which are inherent in the heat trace expansion (as mentioned above the coefficients with odd indices vanish [38], [17]). It is here where the resolvent symbols make their appearance. Let us also mention here an alternative well known formula for the nonzero coefficients,

$$k_{2k}(x) = -\frac{1}{2}\int_{|\xi|=1}\int_{C_1} \log\lambda\,\partial_\lambda^{\frac{n}{2}-j} r_{-2-2k}(x,\xi,\lambda)\,đ\lambda\,đ_S\xi\,. \tag{4.1.29}$$

It is extracted from the ζ-function formulation (4.1.27), equivalent to (4.1.5) yet applicable to a more general class of operators. The inner integral is again over the circle in $\mathbb{C}$ centered at 1 and not enclosing the origin, with $đ\lambda = id\lambda/2\pi$, and the outer integral is over the unit - sphere in T_xM, with rescaled sphere measure $đ_S\xi = d_S\xi/(2\pi)^n$.

Coming back to the the McKean - Singer formula (4.1.10) and the problem of computing the limit (4.1.14) we note that the index (i.e. the left hand side in (4.1.10)) is independent of t whilst for small t the right hand side is approximated in terms of the formal difference

$$\sum_{k\geq 0} c_k\, t^{\frac{k-n}{2}} - \sum_{k\geq 0} \widetilde{c}_k\, t^{\frac{k-n}{2}}$$

where c_k, $\widetilde{c}_k$ are the coefficients in the heat trace expansions of $\triangle$, $\widetilde{\triangle}$ respectively. By letting $t \to 0_+$ the claim is that, by the constancy of the right hand side for arbitrarily small $t > 0$, the limit exists and hence

$$\operatorname{ind} D = c_n - \widetilde{c}_n\,. \tag{4.1.30}$$

Substituting the explicit formulae for the coefficients from the heat trace expansion into (4.1.30), we arrive at

$$\operatorname{ind} D = \int_M \left(\int_{\mathbb{R}^n}\int_\gamma e^{-\lambda}\{\operatorname{tr} r_{-2-2j}(x,\xi,\lambda) - \operatorname{tr}\widetilde{r}_{-2-2j}(x,\xi,\lambda)\}\,đ\lambda đ\xi\right)|dx|\ . \tag{4.1.31}$$

The idea is then to derive the index formula

$$\operatorname{ind} D = \frac{1}{(2\pi)^{n/2}} \int_M \widehat{A}(M) \operatorname{ch}(\mathcal{F})$$

by relating the generating functions for the topological and analytical data described above. This approach has the advantage that it computes the index directly from the first n terms of the local symbols of the resolvent operator. These are polynomials whose coefficients are determined by the local symbol of the Laplacians, together with a finite number of its derivatives. Thus it reflects the local nature of the index quite well. Furthermore, the simplicity of the method itself may be seen as satisfactory, after all the index of an operator is an integer, so in some sense one should be able to determine it via elementary computations.

In Section 4.3 we shall study the technique using as a concrete example the Riemann-Roch-Hirzebruch theorem. Section 4.3.1 sets out the context of the theorem, then in Section 4.3.2 we determine explicit formulae for the resolvent symbols of Laplace operators defined over a Riemann surface, these are then applied to derive the Riemann-Roch formula in Section 4.3.3, again by a direct and elementary calculation. Similar to the previous case the explicit form of our formulae and the method to derive the Riemann Roch theorem are new in the literature.

4.2 Resolvent symbols on closed Riemannian manifolds and heat coefficients

4.2.1 Preliminaries

Let us first recall the essential facts about the resolvent and the heat operator from the viewpoint of pseudodifferential operator theory; for more details we refer to [41]. Let (M, g) be a smooth compact Riemannian manifold of dimension n. The corresponding Laplace - Beltrami operator is locally given by

$$\triangle_g = -\frac{1}{\sqrt{|g|}} \sum_{k,l} \partial_k (g^{kl} \sqrt{|g|}\, \partial_l) \tag{4.2.1}$$

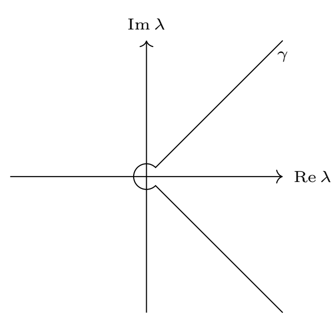
where $\partial_k = \frac{\partial}{\partial x_k}$, $(g^{kl}) = g^{-1}$ denotes the dual metric to g on the cotangent bundle and $|g| = \det g = \det(g_{kl})$. We would like to consider operators of the form $P = \triangle_g + A$ where A denotes a smooth vector field on M (i.e. a smooth section of the tangent bundle), so locally $A = \sum_{k=0}^{n} a_k \partial_k$ with a_k smooth locally supported functions. P is a differential operator of degree 2, certainly elliptic since $\triangle_g$ is elliptic and A (being or degree 1) does not change the principal symbol. We shall restrict our choice of A such that the spectrum of P (Spec(P)) exhibits the same nice properties as the spectrum of $\triangle_g$. In particular, we require it to be discrete and non-negative, accumulating only at infinity

$$0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \ldots \quad \to \infty \tag{4.2.2}$$

and the corresponding smooth eigenfunctions to form a complete orthonormal basis. This is possible provided A is a conservative vector field, which means that it is the gradient of a function; equivalently the differential 1 - form dual to A is exact (cf. [24]). We can then define the heat operator e^{-tP} for $t > 0$ by

$$e^{-tP} := \frac{i}{2\pi} \int_\gamma e^{-t\lambda} (P - \lambda)^{-1} \, d\lambda \tag{4.2.3}$$

where γ is a positively oriented contour, consisting of the rays $\{\, re^{i\pi/4} \mid c \leq r \,\}$ and $\{\, re^{-i\pi/4} \mid c \leq r \,\}$ ($c \in \mathbb{R}$ is small and positive) together with an open arc $\{\, ce^{i\theta} \mid \pi/4 \leq \theta \leq 7\pi/4 \,\}$ round the origin. In particular, γ encloses the spectrum of P (see the sketch below)



The heat operator is smoothing, therefore trace class with

$$\operatorname{Tr}(e^{-tP}) = \int_M k_P(t,x,x)\,|dx| \tag{4.2.4}$$

where $k_P(t,x,y)$ denotes the Schwartz kernel of e^{-tP}, and $|dx|$ locally identifies with Lebesgue measure.

If $\lambda \notin \operatorname{Spec}(P)$ then $(P-\lambda)$ is elliptic, i.e. invertible. The resolvent $(P-\lambda)^{-1}$ is a pseudodifferential operator, hence it can be represented by its distributional Schwartz kernel k_λ. Locally the latter is given, modulo smoothing operators (i.e. operators of arbitrarily low order) by an oscillatory integral

$$k_\lambda(x,y) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, r(x,\xi,\lambda)\, đ\xi \tag{4.2.5}$$

where $r(x,\xi,\lambda)$ is the local symbol of the resolvent operator and $đ\xi = (2\pi)^{-n}d\xi$ denotes (rescaled) Lebesgue measure. The symbol $r(x,\xi,\lambda)$ admits an asymptotic expansion

$$r(x,\xi,\lambda) \quad \backsim \quad \sum_{j\geq 0} r_{-2-j}(x,\xi,\lambda) \qquad \left(|\xi| + |\lambda|^{1/2} \geq 1\right) \tag{4.2.6}$$

where each term $r_{-2-j}(x,\xi,\lambda)$ in (4.2.6) is quasi - homogeneous in (ξ,λ) of degree $-2-j$, meaning that $r_{-2-j}(x,t\xi,t^2\lambda) = t^{-2-j}r_{-2-j}(x,\xi,\lambda)$ for $t>0$, $|\xi| + |\lambda|^{1/2} \geq 1$. This asymptotic expansion arises in the construction of the parametrix for $(P-\lambda)^{-1}$; in this process one determines local symbols $r(x,y,\lambda)$ such that the operator B obtained by patching together the local operators $\mathsf{Op}[r]$ (the operator whose Schwartz kernel is defined by the symbol $r(x,y,\lambda)$) satisfies

$$(P-I)\circ B = I + R_1 \quad \text{and} \quad B\circ(P-I) = I + R_2$$

where R_1, R_2 are smoothing operators (i.e. operators of arbitrarily low order) and the product $\circ$ here is operator composition. Slightly more concretely, let σ_P denote the local symbol of P; then from the symbol calculus we know that the product $\sigma_P \cdot r$ (pointwise multiplication of functions or, more generally, matrices) is a classical parameter dependent symbol of order zero with asymptotic expansion

$$(\sigma_P \cdot r)(x, \xi, \lambda) \backsim 1\,. \tag{4.2.7}$$

On the other hand, if we formally write down an asymptotic expansion $r(x, \xi, \lambda) \backsim \sum_{j\geq 0} r_{-2-j}(x, \xi, \lambda)$ for the symbol of the resolvent and apply the composition formula (c.f. [41, Theorem 3.4]) we obtain

$$(\sigma_P \cdot r)(x, \xi, \lambda) \backsim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_P(x, \xi) D_x^\alpha r(x, \xi, \lambda) \tag{4.2.8}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ ranges over all possible multi - indices, $\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n}$ with $\partial_{u_i}^{\alpha_i} = \partial^{\alpha_i}/\partial u_i^{\alpha_i}$ and $D_x^\alpha = (-i)^{|\alpha|} \partial_x^{\alpha_i}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n$. By comparing the terms of common homogeneity in these expansions one obtains the following well - known definition of the resolvent symbols (see for example [14]):

$$r_{-2} = (a_2 - \lambda)^{-1}\,, \tag{4.2.9}$$

$$r_{-2-j} = -r_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!} (\partial_\xi^\mu a_{2-k})(D_x^\mu r_{-2-l}) \qquad (j \geq 1) \tag{4.2.10}$$

where a_j denotes the term in the local symbol σ_P that is homogeneous of order j and μ denotes a multi-index. In particular we see that the functions r_{-2-j} are polynomials whose coefficients are determined by the local symbol of the operator P together with a finite number of its derivatives. Our aim in the next section is to turn these recursive formulae into concrete polynomial expressions for the resolvent symbols.

4.2.2 Explicit formulae for the resolvent symbols

Expanding the operator P in local coordinates we obtain

$$P = \sum_{k,l} g^{kl}(-i\partial_k)(-i\partial_l) + \sum_k \Big(\sum_l (2|g|)^{-1} g^{kl}(-i\partial_l |g|) \\ + (-i\partial_l\, g^{kl}) + i a_k \Big)(-i\partial_k)\,.$$

(We recall here that $|g| = \det g$). If we replace $(-i\partial_k)$ by ξ_k we obtain the local symbol

$$\sigma_P(x,\xi) = |\xi|_g^2 + \sum_k b_k(x)\,\xi_k \tag{4.2.11}$$

with

$$|\xi|_g^2 = \sum_{k,l} g^{kl}\,\xi_k\xi_l \tag{4.2.12}$$

and

$$b_k = \sum_l \frac{1}{2}|g|^{-1}g^{kl}D_{x_l}|g| + D_{x_l}\,g^{kl} + ia_k\,. \tag{4.2.13}$$

We shall also use the notation

$$\sigma_P = a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi) \tag{4.2.14}$$

where the term a_k is homogeneous of degree k in ξ; that is

$$a_2(x,\xi) = |\xi|_g^2, \qquad a_1(x,\xi) = \sum_k b_k(x)\,\xi_k, \qquad a_0(x,\xi) = 0. \tag{4.2.15}$$

We can now state and prove the main theorem of this section:

Theorem 4.2.1. *The first three resolvent symbols as defined in* (4.2.10) *have the following explicit representations as polynomials in ξ:*

$$r_{-2} = (|\xi|_g^2 - \lambda)^{-1} \tag{4.2.16}$$

$$r_{-3} = 2r_{-2}^3 \sum_{l,s,p,q} g^{sl}(D_{x_l}g^{pq})\,\xi_s\xi_p\xi_q - r_{-2}^2\sum_l b_l\xi_l \tag{4.2.17}$$

and for the third resolvent symbol we have

$$\begin{aligned} r_{-4} = 12r_{-2}^5 &\sum_{l,i,j,s,p,q,k,t} g^{tk}g^{sl}(D_{x_l}g^{pq})(D_{x_k}g^{ij})\,\xi_i\xi_j\xi_s\xi_p\xi_q\xi_t \\ &- 2r_{-2}^4\sum_{k,p,q,s,t} g^{kk}(D_{x_k}g^{pq})(D_{x_k}g^{st})\,\xi_p\xi_q\xi_s\xi_t + r_{-2}^3\sum_{k,s,t} g^{kk}(D_{x_k}^2 g^{st})\,\xi_s\xi_t \\ &- 4r_{-2}^4\sum_{\substack{k,l,p,q,s,t\\ k\neq l}} g^{kl}(D_{x_l}g^{pq})(D_{x_k}g^{st})\,\xi_p\xi_q\xi_s\xi_t \;+\; 2r_{-2}^3\sum_{\substack{k,l,s,t\\ k\neq l}} g^{kl}(D_{x_l,x_k}^2 g^{st})\,\xi_s\xi_t \end{aligned}$$

$$- 4r_{-2}^4 \sum_{l,s,p,q,k,t} g^{tk}(D_{x_k} g^{sl})(D_{x_l} g^{pq})\, \xi_s \xi_p \xi_q \xi_t - 4r_{-2}^4 \sum_{l,s,p,q,k,t} g^{tk} g^{sl} (D^2_{x_k,x_l} g^{pq})\, \xi_s \xi_p \xi_q \xi_t$$

$$- 6r_{-2}^4 \sum_{l,i,j,k,t} b_l\, g^{tk}(D_{x_k} g^{ij}) \xi_i \xi_j \xi_l \xi_t + 2r_{-2}^3 \sum_{l,t,k} g^{tk}(D_{x_k} b_l) \xi_l \xi_t + r_{-2}^3 \sum_{l,s,t} b_l (D_{x_l} g^{st})\, \xi_s \xi_t$$

$$+ r_{-2}^3 \sum_{k,l} b_k b_l\, \xi_k \xi_l \tag{4.2.18}$$

where

$$D_{x_k} b_j = \sum_l \frac{1}{2} |g|^{-1} \Big((D_{x_k} g^{jl})(Dx_l |g|) + g^{jl}(D^2_{x_k,x_l} |g|) - |g|^{-1} g^{jl} (D_{x_k}|g|)(D_{x_l}|g|) + 2|g| D^2_{x_k,x_l} g^{jl} \Big) + i D_{x_k} a_j \,. \tag{4.2.19}$$

Proof. First, the term r_{-2} is immediate from (4.2.9). Next, from (4.2.10) we get

$$r_{-3} = -r_{-2} \sum_{\substack{|\mu|+k+l=1 \\ l<1}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l} \,. \tag{4.2.20}$$

The condition $l < 1$ implies $l = 0$ throughout, and the condition $|\mu| + k + l = 1$ forces $\mu! = 1$ in all summands. This simplifies the above expression to

$$r_{-3} = -r_{-2} \sum_{|\mu|+k=1} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2} \,. \tag{4.2.21}$$

We expand into two summands

$$\begin{aligned} r_{-3} &= -r_{-2} \underbrace{\left(\sum_{|\mu|=1} \partial_\xi^\mu a_2 D_x^\mu r_{-2} \right)}_{|\mu|=1,k=0} - r_{-2} \underbrace{\left(a_1 r_{-2} \right)}_{|\mu|=0,k=1} \\ &= -r_{-2} \underbrace{\left(\sum_l \partial_{\xi_l} a_2 D_{x_l} r_{-2} \right)}_{|\mu|=1,k=0} - r_{-2} \underbrace{\left(\sum_l b_l \xi_l r_{-2} \right)}_{|\mu|=0,k=1} \,. \end{aligned} \tag{4.2.22}$$

For the first term note that

$$\partial_{\xi_l} a_2 = \partial_{\xi_l} \left(\sum_{s,t} g^{st}\, \xi_s \xi_t \right) = 2 \sum_s g^{sl} \xi_s \quad \text{(using the symmetry } g^{sl} = g^{ls} \text{)} \tag{4.2.23}$$

and for any positive integer m, we have

$$D_{x_l} r_{-2}^m = D_{x_l}(a_2-\lambda)^{-m} = -m\, r_{-2}^{m+1} \sum_{s,t} (D_{x_l} g^{st})\, \xi_s \xi_t \,. \tag{4.2.24}$$

Substitution of (4.2.23) and (4.2.24) into (4.2.22) gives

$$\begin{aligned}
r_{-3} &= -r_{-2} \underbrace{\left(\sum_l \partial_{\xi_l} a_2 D_{x_l} r_{-2} \right)}_{|\mu|=1,k=0} - r_{-2} \underbrace{\left(\sum_l b_l \xi_l r_{-2} \right)}_{|\mu|=0,k=1} \\
&= -r_{-2} \underbrace{\left(\sum_l \left(2 \sum_s g^{sl} \xi_s \right) \left(-r_{-2}^2 \sum_{p,q} (D_{x_l} g^{pq})\, \xi_p \xi_q \right) \right)}_{|\mu|=1,k=0} - r_{-2} \underbrace{\left(\sum_l b_l \xi_l r_{-2} \right)}_{|\mu|=0,k=1} \\
&= -r_{-2} \underbrace{\left(-2 r_{-2}^2 \sum_{l,s,p,q} g^{sl} (D_{x_l} g^{pq})\, \xi_s \xi_p \xi_q \right)}_{|\mu|=1,k=0} - r_{-2} \underbrace{\left(r_{-2} \sum_l b_l \xi_l \right)}_{|\mu|=0,k=1} \\
&= 2 r_{-2}^3 \sum_{l,s,p,q} g^{sl} (D_{x_l} g^{pq})\, \xi_s \xi_p \xi_q - r_{-2}^2 \sum_l b_l \xi_l \,.
\end{aligned} \tag{4.2.25}$$

which is the right hand side in equation (4.2.17).

The final formula to deduce is that for r_{-4}; we start from (4.2.10) with the expression

$$r_{-4} = -r_{-2} \sum_{\substack{|\mu|+k+l=2 \\ l<2}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l} \,. \tag{4.2.26}$$

Here the condition $l<2$ implies $l=0$ or $l=1$. Thus we can break up the above into

$$r_{-4} = \underbrace{-r_{-2} \sum_{|\mu|+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2}}_{l=0} \quad \underbrace{-r_{-2} \sum_{|\mu|+k=1} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-3}}_{l=1} \tag{4.2.27}$$

and repeat this process by letting $k=0,1$ or 2 in the first summand and $k=0$ or 1 in the second summand. We shall proceed consecutively for the the case $l=0$ and $l=1$.

Term corresponding to $l = 0$: Fully expanding this term gives

$$
\begin{aligned}
&- r_{-2} \sum_{|\mu|+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2} \\
&= \underbrace{-r_{-2} \sum_{|\mu|=2} \frac{1}{\mu!} \partial_\xi^\mu a_2 D_x^\mu r_{-2}}_{k=0} \quad \underbrace{-r_{-2} \sum_{|\mu|=1} \frac{1}{\mu!} \partial_\xi^\mu a_1 D_x^\mu r_{-2}}_{k=1}
\end{aligned}
\tag{4.2.28}
$$

where we use $a_0 = 0$, so the last term that comes from $k = 2$ vanishes. For the first term we need to compute $\partial_\xi^\mu a_2$ as well as $D_x^\mu r_{-2}$ with $|\mu| = 2$, that is $\partial_\xi^\mu a_2 = \partial_{\xi_k} (\partial_{\xi_l} a_2)$ and likewise $D_x^\mu r_{-2} = D_{x_l} (D_{x_k} r_{-2})$. We already computed the first derivative in (4.2.23) and (4.2.24); substituting these we get

$$
\partial_{\xi_k} (\partial_{\xi_l} a_2) = \partial_{\xi_k} \left(2 \sum_s g^{sl} \xi_s \right) = 2 \sum_s g^{sl} (\partial_{\xi_k} \xi_s) = 2 g^{kl} \tag{4.2.29}
$$

and

$$
\begin{aligned}
&D_{x_l} (D_{x_k} r_{-2}) = D_{x_l} \left(-r_{-2}^2 \sum_{s,t} (D_{x_k} g^{st})\, \xi_s \xi_t \right) \\
&= - \underbrace{(D_{x_l} r_{-2}^2)}_{(*)} \sum_{s,t} (D_{x_k} g^{st})\, \xi_s \xi_t \quad - r_{-2}^2 \sum_{s,t} (D^2_{x_l, x_k} g^{st})\, \xi_s \xi_t \,.
\end{aligned}
$$

Then substitution of (4.2.24) into the term $(*)$ evaluates the above expression to

$$
\begin{aligned}
&= - \left(- 2 r_{-2}^3 \sum_{p,q} (D_{x_l} g^{pq})\, \xi_p \xi_q \right) \sum_{s,t} (D_{x_k} g^{st})\, \xi_s \xi_t \quad - r_{-2}^2 \sum_{s,t} (D^2_{x_l, x_k} g^{st})\, \xi_s \xi_t \\
&= 2 r_{-2}^3 \sum_{p,q,s,t} (D_{x_l} g^{pq})(D_{x_k} g^{st})\, \xi_p \xi_q \xi_s \xi_t \quad - r_{-2}^2 \sum_{s,t} (D^2_{x_l, x_k} g^{st})\, \xi_s \xi_t \,.
\end{aligned}
\tag{4.2.30}
$$

Combining these results we see that the first summand in (4.2.28) expands into

$$
\begin{aligned}
&- 2 r_{-2}^4 \sum_{k,p,q,s,t} g^{kk} (D_{x_k} g^{pq})(D_{x_k} g^{st})\, \xi_p \xi_q \xi_s \xi_t \;+\; r_{-2}^3 \sum_{k,s,t} g^{kk} (D^2_{x_k} g^{st})\, \xi_s \xi_t \\
&\quad - 4 r_{-2}^4 \sum_{\substack{k,l,p,q,s,t \\ k \neq l}} g^{kl} (D_{x_l} g^{pq})(D_{x_k} g^{st})\, \xi_p \xi_q \xi_s \xi_t \;+\; 2 r_{-2}^3 \sum_{\substack{k,l,s,t \\ k \neq l}} g^{kl} (D^2_{x_l, x_k} g^{st})\, \xi_s \xi_t \,.
\end{aligned}
\tag{4.2.31}
$$

Now for the second term in (4.2.28), that is

$$- r_{-2} \sum_{|\mu|=1} \frac{1}{\mu!} \partial_\xi^\mu a_1 D_x^\mu r_{-2} = -r_{-2} \sum_l \partial_{\xi_l} a_1 D_{x_l} r_{-2} \tag{4.2.32}$$

we note that

$$\partial_{\xi_l} a_1 = \partial_{\xi_l} \left(\sum_s b_s \, \xi_s \right) = b_l \tag{4.2.33}$$

and (from (4.2.24))

$$D_{x_l} r_{-2} = -r_{-2}^2 \sum_{s,t} (D_{x_l} g^{st}) \, \xi_s \xi_t \,. \tag{4.2.34}$$

Insertion of these two expressions into (4.2.32) gives

$$- r_{-2} \sum_{|\mu|=1} \frac{1}{\mu!} \partial_\xi^\mu a_1 D_x^\mu r_{-2} = r_{-2}^3 \sum_{l,s,t} b_l (D_{x_l} g^{st}) \, \xi_s \xi_t \,. \tag{4.2.35}$$

Using the lines (4.2.31) and (4.2.35) we can now fully expand equation (4.2.28) :

$$\begin{aligned}
& - r_{-2} \sum_{|\mu|+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2} \\
& = -2 r_{-2}^4 \sum_{k,p,q,s,t} g^{kk} (D_{x_k} g^{pq})(D_{x_k} g^{st}) \, \xi_p \xi_q \xi_s \xi_t \; + \; r_{-2}^3 \sum_{k,s,t} g^{kk} (D_{x_k}^2 g^{st}) \, \xi_s \xi_t \\
& \quad - 4 r_{-2}^4 \sum_{\substack{k,l,p,q,s,t \\ k \neq l}} g^{kl} (D_{x_l} g^{pq})(D_{x_k} g^{st}) \, \xi_p \xi_q \xi_s \xi_t \; + \; 2 r_{-2}^3 \sum_{\substack{k,l,s,t \\ k \neq l}} g^{kl} (D_{x_l, x_k}^2 g^{st}) \, \xi_s \xi_t \\
& \quad + r_{-2}^3 \sum_{l,s,t} b_l (D_{x_l} g^{st}) \, \xi_s \xi_t
\end{aligned} \tag{4.2.36}$$

which establishes an explicit version for the first term of (4.2.27).

Term corresponding to $l = 1$: Here we need to determine the right hand side of

$$- r_{-2} \sum_{|\mu|+k=1} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-3} = -r_{-2} \sum_k \partial_{\xi_k} a_2 \underbrace{D_{x_k} r_{-3}}_{(**)} \quad -r_{-2} (a_1 r_{-3}) \,. \tag{4.2.37}$$

We already computed $\partial_{\xi_k} a_2$ in (4.2.23) and the resolvent symbol r_{-3} is also established. Let us list these and the explicit form of a_1 here for convenience:

$$a_1 = \sum_k b_k \, \xi_k \tag{4.2.38}$$

$$\partial_{\xi_l} a_2 = 2 \sum_s g^{sl} \xi_s \tag{4.2.39}$$

$$r_{-3} = 2 r_{-2}^3 \sum_{l,s,p,q} g^{sl} (D_{x_l} g^{pq}) \, \xi_s \xi_p \xi_q - r_{-2}^2 \sum_l b_l \xi_l \, . \tag{4.2.40}$$

The only part that remains to be looked at is $(**)$ in (4.2.37); which gives

$$\begin{aligned}
D_{x_k} r_{-3} &= D_{x_k} \left(2 r_{-2}^3 \sum_{l,s,p,q} g^{sl} (D_{x_l} g^{pq}) \, \xi_s \xi_p \xi_q - r_{-2}^2 \sum_l b_l \xi_l \right) \\
&= -6 r_{-2}^4 \sum_{l,i,j,s,p,q} g^{sl} (D_{x_l} g^{pq}) (D_{x_k} g^{ij}) \, \xi_i \xi_j \xi_s \xi_p \xi_q \\
&\quad + 2 r_{-2}^3 \sum_{l,s,p,q} (D_{x_k} g^{sl}) (D_{x_l} g^{pq}) \, \xi_s \xi_p \xi_q + 2 r_{-2}^3 \sum_{l,s,p,q} g^{sl} (D^2_{x_k, x_l} g^{pq}) \, \xi_s \xi_p \xi_q \\
&\quad + 2 r_{-2}^3 \sum_{l,s,t} b_l (D_{x_k} g^{st}) \xi_s \xi_t \xi_l - r_{-2}^2 \sum_l (D_{x_k} b_l) \xi_l \, .
\end{aligned} \tag{4.2.41}$$

Thus in summary we obtain the following for (4.2.37) :

$$\begin{aligned}
&- r_{-2} \sum_{|\mu|+k=1} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-3} \\
&= 12 r_{-2}^5 \sum_{l,i,j,s,p,q,k,t} g^{tk} g^{sl} (D_{x_l} g^{pq}) (D_{x_k} g^{ij}) \, \xi_i \xi_j \xi_s \xi_p \xi_q \xi_t \\
&\quad - 4 r_{-2}^4 \sum_{l,s,p,q,k,t} g^{tk} (D_{x_k} g^{sl}) (D_{x_l} g^{pq}) \, \xi_s \xi_p \xi_q \xi_t \\
&\quad - 4 r_{-2}^4 \sum_{l,s,p,q,k,t} g^{tk} g^{sl} (D^2_{x_k, x_l} g^{pq}) \, \xi_s \xi_p \xi_q \xi_t \\
&\quad - 4 r_{-2}^4 \sum_{l,i,j,k,t} b_l \, g^{tk} (D_{x_k} g^{ij}) \xi_i \xi_j \xi_l \xi_t + 2 r_{-2}^3 \sum_{l,t,k} g^{tk} (D_{x_k} b_l) \xi_l \xi_t \\
&\quad - 2 r_{-2}^4 \sum_{k,l,s,p,q} b_k \, g^{sl} (D_{x_l} g^{pq}) \, \xi_s \xi_p \xi_q \xi_k + r_{-2}^3 \sum_{k,l} b_k b_l \, \xi_k \xi_l \, .
\end{aligned} \tag{4.2.42}$$

Full expression Finally we concatenate (4.2.36) and (4.2.42) in order to determine the complete expression for r_{-4}. Rearranging and collecting like terms, we obtain

$$
\begin{aligned}
r_{-4} &= \underbrace{-r_{-2}\sum_{|\mu|+k=2}\frac{1}{\mu!}\partial_\xi^\mu a_{2-k}D_x^\mu r_{-2}}_{\text{equation (4.2.36)}} \quad \underbrace{-r_{-2}\sum_{|\mu|+k=1}\frac{1}{\mu!}\partial_\xi^\mu a_{2-k}D_x^\mu r_{-3}}_{\text{equation (4.2.42)}} \\
&= 12r_{-2}^5\sum_{l,i,j,s,p,q,k,t} g^{tk}g^{sl}(D_{x_l}g^{pq})(D_{x_k}g^{ij})\,\xi_i\xi_j\xi_s\xi_p\xi_q\xi_t \\
&\quad - 2r_{-2}^4\sum_{k,p,q,s,t} g^{kk}(D_{x_k}g^{pq})(D_{x_k}g^{st})\,\xi_p\xi_q\xi_s\xi_t \\
&\quad + r_{-2}^3\sum_{k,s,t} g^{kk}(D_{x_k}^2 g^{st})\,\xi_s\xi_t - 4r_{-2}^4\sum_{\substack{k,l,p,q,s,t\\ k\neq l}} g^{kl}(D_{x_l}g^{pq})(D_{x_k}g^{st})\,\xi_p\xi_q\xi_s\xi_t \\
&\quad + 2r_{-2}^3\sum_{\substack{k,l,s,t\\ k\neq l}} g^{kl}(D_{x_l,x_k}^2 g^{st})\,\xi_s\xi_t - 4r_{-2}^4\sum_{l,s,p,q,k,t} g^{tk}(D_{x_k}g^{sl})(D_{x_l}g^{pq})\,\xi_s\xi_p\xi_q\xi_t \\
&\quad - 4r_{-2}^4\sum_{l,s,p,q,k,t} g^{tk}g^{sl}(D_{x_k,x_l}^2 g^{pq})\,\xi_s\xi_p\xi_q\xi_t - 6r_{-2}^4\sum_{l,i,j,k,t} b_l\, g^{tk}(D_{x_k}g^{ij})\xi_i\xi_j\xi_l\xi_t \\
&\quad + 2r_{-2}^3\sum_{l,t,k} g^{tk}(D_{x_k}b_l)\xi_l\xi_t + r_{-2}^3\sum_{l,s,t} b_l(D_{x_l}g^{st})\,\xi_s\xi_t + r_{-2}^3\sum_{k,l} b_k b_l\,\xi_k\xi_l
\end{aligned}
$$

as required. □

4.2.3 The first three heat coefficients

Now that we have access to the closed formulas above let us use the resolvent symbols to directly derive the first three heat coefficients $c_{\frac{-n}{2}}, c_{\frac{-n+1}{2}}$ and $c_{\frac{-n+2}{2}}$ in the short time asymptotic expansion of the heat trace:

$$\operatorname{Tr}\left(e^{-tP}\right) \backsim_{t\to 0_+} \sum_{j\geq 0} c_{\frac{-n+j}{2}}\, t^{\frac{-n+j}{2}} \tag{4.2.43}$$

(we refer to [15], [14] or alternatively [36] for an account of the existence and derivation of the asymptotic expansion). Here $n = \dim(M)$ and

$$c_{\frac{-n+j}{2}} = \int_M c_{\frac{-n+j}{2}}(x)\,|dx| \tag{4.2.44}$$

with

$$c_{\frac{-n+j}{2}}(x)|dx| = \int_{\mathbb{R}^n} \left(\frac{i}{2\pi} \int_\gamma e^{-\lambda}\, r_{-2-j}(x,\xi,\lambda)\, d\lambda \right) đ\xi \; dx\,. \tag{4.2.45}$$

(the notation dx respectively $đ\xi = (2\pi)^{-n}\, d\xi$ denotes Lebesgue measure, the latter rescaled). Finding these coefficients concretely is interesting from the geometric point of view, in particular the following identities are well - known ([24]):

$$c_{\frac{-n}{2}} = \frac{1}{4\pi^{\frac{n}{2}}} \mathrm{Vol}_g(M) \tag{4.2.46}$$

and

$$c_{\frac{-n+2}{2}} = \frac{1}{(4\pi)^{\frac{n}{2}}} \left(\frac{1}{3} \int_M K\, d\mu_g + \frac{1}{2} \int_M \mathrm{div}(A)\, d\mu_g - \frac{1}{4} \int_M |A|^2\, d\mu_g \right) \tag{4.2.47}$$

where $d\mu_g$ denotes the volume form induced by the metric g, K is the scalar curvature and $\mathrm{div}(A)$, $|A|^2$ denote the (Riemannian) divergence, and length of A, respectively. Whereas the common approach to their derivation uses global estimates on the heat kernel (as shown in [24]) or abstract invariance theory (c.f.[14]), we shall derive these now *directly*, that is using only the local data from the resolvent symbols via the formulas in Theorem 4.2.1.

To start let us note some general properties of the involved integrals. First, by the Cauchy Residue theorem we have

$$\frac{i}{2\pi} \int_\gamma e^{-\lambda}\, r_{-2}^k\, d\lambda = \frac{i}{2\pi} \int_\gamma \frac{e^{-\lambda}}{(|\xi|_g^2 - \lambda)^k} d\lambda = \frac{1}{(k-1)!} e^{-|\xi|_g^2} \tag{4.2.48}$$

for any $k \geq 1$. Secondly, for a real positive definite symmetric matrix $A = (A_{ij})$ and a polynomial $p(x)$ in $x \in \mathbb{R}^k$ we have the Gaussian Integral (cf [44])

$$\int_{\mathbb{R}^k} p(x) \exp\left(-\frac{1}{2} \sum_{i,j}^n A_{ij}\, x_i x_j \right) dx = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\mathrm{det} A}} \exp\left(\frac{1}{2} \sum_{i,j}^n (A_{ij})^{-1} \partial_{x_i} \partial_{x_j} \right) p(x) \Big|_{x=0} . \tag{4.2.49}$$

As we shall see in a moment we need to determine integrals of the form

$$\int_{\mathbb{R}^n} \xi^\alpha \, \exp\left(- \sum_{k,l}^n g^{kl} \xi_k \xi_l \right) d\xi \tag{4.2.50}$$

(here, $\alpha = (\alpha_1, \ldots \alpha_n)$ is a multi-index, so that $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi^{\alpha_n}$ denotes a monomial in $\xi \in \mathbb{R}^n$). Substituting (4.2.49) into (4.2.50) yields

$$\begin{aligned} \int_{\mathbb{R}^n} \xi^\alpha \exp\left(-\sum_{k,l}^n g^{kl}\xi_k\xi_l\right) d\xi &= \int_{\mathbb{R}^n} \xi^\alpha \exp\left(-\frac{1}{2}\sum_{k,l}^n 2g^{kl}\xi_k\xi_l\right) d\xi \\ &= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(2g^{-1})}} \exp\left(\frac{1}{4}\sum_{k,l}^n g_{kl}\partial_{\xi_k}\partial_{\xi_l}\right) \xi^\alpha\Big|_{\xi=0} \\ &= \pi^{\frac{n}{2}}\sqrt{\det g}\, \exp\left(\frac{1}{4}\sum_{k,l}^n g_{kl}\partial_{\xi_k}\partial_{\xi_l}\right) \xi^\alpha\Big|_{\xi=0}. \end{aligned} \tag{4.2.51}$$

Note that any monomial ξ^α of odd degree (that is with $|\alpha| = 2k+1$ for some integer k) evaluates to zero in (4.2.51) because in this case the polynomial

$$\exp\left(\frac{1}{4}\sum_{k,l}^n g_{kl}\partial_{\xi_k}\partial_{\xi_l}\right) \xi^\alpha \tag{4.2.52}$$

has no constant term. Hence, evaluation at $\xi = 0$ equates it to zero.

For the final observation let us choose normal coordinates on our manifold M centered at the point $p \in M$, say. Then the metric tensor g evaluates at p to the identity, that is $\sum_{i,j} g^{ij}\xi_i\xi_j = \sum_k \xi_k^2 = |\xi|^2$ and we can evaluate (4.2.50) via Fubini's theorem as a simple product of Gaussian integrals over the real line:

$$\int_{\mathbb{R}^n} \xi^\alpha \exp\left(-\sum_{k,l}^n \delta^{kl}\xi_k\xi_l\right) d\xi = \int_{\mathbb{R}^n} \xi^\alpha\, e^{-|\xi|^2}\, d\xi = \prod_{i=1}^n \int_{\mathbb{R}} \xi_i^{\alpha_i}\, e^{-\xi_i^2}\, d\xi_i\,. \tag{4.2.53}$$

Also, for a positive integer k and any real number $\beta > 0$ we have

$$\int_{\mathbb{R}} x^k\, e^{-\beta x^2}\, dx = \begin{cases} \dfrac{\Gamma(k+\frac{1}{2})}{\beta^{k+\frac{1}{2}}} & \text{if k is even} \\ 0 & \text{if k is odd} \end{cases} \tag{4.2.54}$$

where $\Gamma(z)$ denotes the Gamma function (the first case can be deduced using the change of variable $y = \beta x^2$ and the latter case follows from repeated integration by parts). Here the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\, dt \quad (\mathrm{Re}(z) > 0), \tag{4.2.55}$$

we shall also need the fact that It satisfies the equations

$$\Gamma(z+1) = z\,\Gamma(z) \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi}. \tag{4.2.56}$$

From (4.2.54) it therefore follows that (4.2.57) evaluates to

$$\prod_{i=1}^{n} \int_{\mathbb{R}} \xi_i^{\alpha_i}\, e^{-\xi_i^2}\, d\xi_i = \begin{cases} \prod_{i=1}^{n} \Gamma(\alpha_i + \frac{1}{2}) & \text{all } \alpha_i \text{ are even} \\ 0 & \text{otherwise} \end{cases}. \tag{4.2.57}$$

We are now ready to deduce the heat coefficients.

The heat coefficient $c_{-\frac{n}{2}}$ For this we evaluate the integral

$$c_{-\frac{n}{2}}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\frac{i}{2\pi} \int_{\Gamma} e^{-\lambda}\, r_{-2}(x,\xi,\lambda)\, d\lambda \right) d\xi\,, \tag{4.2.58}$$

this is by (4.2.48) equal to

$$\begin{aligned} &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-|\xi|_g^2}\, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \sum_{k,l}^{n} 2g^{kl}\xi_k\xi_l \right) d\xi \end{aligned} \tag{4.2.59}$$

Now (4.2.59) is is an integral of the form (4.2.50) with $|\alpha| = 0$; hence the above expression reduces to

$$\begin{aligned} &= \frac{1}{(2\pi)^n} \left(\pi^{\frac{n}{2}} \sqrt{\det g} \right) \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}} \sqrt{\det g} \end{aligned} \tag{4.2.60}$$

and we therefore see from (4.2.44) that

$$c_{\frac{-n+j}{2}} = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_M \sqrt{\det g}\, |dx| = \mathrm{Vol}_g(M)\,. \tag{4.2.61}$$

The heat coefficient $c_{\frac{-n+1}{2}}$ Here the expression under consideration becomes

$$c_{\frac{-n+1}{2}}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\frac{i}{(2\pi)} \int_{\gamma} e^{-\lambda}\, r_{-3}(x,\xi,\lambda)\, d\lambda \right) d\xi\,, \tag{4.2.62}$$

but from (4.2.17) one can immediately deduce (using the remark below equation (4.2.51)) that the above expression evaluates to zero since all the summands in r_{-3} are odd monomials in ξ. So

$$c_{\frac{-n+1}{2}} = \frac{1}{(2\pi)^n} \int_M c_{\frac{-n+1}{2}}(x)\,|dx| = 0\,. \tag{4.2.63}$$

Remark 4.2.2. Let us note here that the odd coefficients vanish in general as one can see by making a change of variable $\xi \mapsto -\xi$ in (4.2.45) and using the quasi - homogeneity of the resolvent symbol.

The heat coefficient $c_{\frac{-n+2}{2}}$ To proceed with the third coefficient we introduce normal coordinates in a neighbourhood of a point $p \in M$. Thus on a small patch centered at p, the metric is approximated by

$$g_{ij} = \delta_{ij} + \frac{1}{3}\sum_{k,l} R_{ikjl}\, x_k x_l + O(|x|^3) \tag{4.2.64}$$

for x close enough to p, where R_{ikjl} denotes the components of the Riemann curvature tensor associated to g and δ_{ij} is the Kroenecker delta (c.f. [38, Section 3.5.3.3]). Likewise for the inverse metric (for x close to the point p) we have

$$g^{ij} = \delta_{ij} - \frac{1}{3}\sum_{k,l} R_{ikjl}\, x_k x_l + O(|x|^3)\,. \tag{4.2.65}$$

In particular, when evaluated at p (i.e. where $x = 0$), g is the identity and the first partial derivatives vanish. In view of (4.2.13) the first order part of P therefore reduces at the point p to $b_k = ia_k$, moreover

$$|\xi|_g^2 = |\xi|^2 = \sum_k \xi^2\,. \tag{4.2.66}$$

Also, one can see from (4.2.65) that the second partial derivatives of the components of g (evaluated at p) are expressions in terms of the Riemann curvature tensor. Using the simplifications in this coordinate system the closed formula for r_{-4} reduces to

$$r_{-4} = r_{-2}^3 \sum_{k,s,t} (D_{x_k}^2 g^{st})\,\xi_s\xi_t - 4r_{-2}^4 \sum_{l,p,q,k} (D_{x_k,x_l}^2 g^{pq})\,\xi_l\xi_p\xi_q\xi_k$$

$$+ 2r_{-2}^3 \sum_{l,k} (D_{x_k} b_l)\, \xi_l \xi_k - r_{-2}^3 \sum_{k,l} a_k a_l\, \xi_k \xi_l \,. \tag{4.2.67}$$

Now we substitute this into the integrand of the heat coefficient:

$$c_{\frac{-n+2}{2}}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\frac{i}{2\pi} \int_\gamma e^{-\lambda}\, r_{-4}(x,\xi,\lambda)\, d\lambda \right) d\xi \tag{4.2.68}$$

and obtain

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{i}{2\pi} \int_\gamma e^{-\lambda} \Bigg(r_{-2}^3 \sum_{k,s,t} (D_{x_k}^2 g^{st})\, \xi_s \xi_t - 4 r_{-2}^4 \sum_{l,p,q,k} (D_{x_k,x_l}^2 g^{pq})\, \xi_l \xi_p \xi_q \xi_k$$
$$+ 2r_{-2}^3 \sum_{l,k} (D_{x_k} b_l)\, \xi_l \xi_k - r_{-2}^3 \sum_{k,l} a_k a_l\, \xi_k \xi_l \Bigg) d\lambda\, d\xi \,.$$

Evaluating the contour integral by use of (4.2.48) and (4.2.66) simplifies the expression to

$$= \frac{1}{2} \sum_{k,s,t} (D_{x_k}^2 g^{st}) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_s \xi_t\, e^{-|\xi|^2}\, d\xi \right) \tag{4.2.69}$$

$$- \frac{2}{3} \sum_{l,p,q,k} (D_{x_k,x_l}^2 g^{pq}) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_l \xi_p \xi_q \xi_k\, e^{-|\xi|^2}\, d\xi \right) \tag{4.2.70}$$

$$+ \sum_{l,k} (D_{x_k} b_l) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_l \xi_k\, e^{-|\xi|^2}\, d\xi \right) \tag{4.2.71}$$

$$- \frac{1}{2} \sum_{k,l} a_k a_l \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_k \xi_l\, e^{-|\xi|^2}\, d\xi \right) \tag{4.2.72}$$

Note that, by equation (4.2.57) we can ignore any terms that contain odd powers of ξ_i, since those will integrate to zero. This means that term (4.2.69) evaluates to

$$\frac{1}{2} \sum_{k,s,t} (D_{x_k}^2 g^{st}) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_s \xi_t\, e^{-|\xi|^2}\, d\xi \right)$$
$$= \frac{1}{2} \sum_{k,s} (D_{x_k}^2 g^{ss}) \left(\frac{1}{2\pi} \Gamma(\frac{3}{2}) \right) \prod_{k \neq s} \frac{1}{2\pi} \Gamma(\frac{1}{2})$$

using also (4.2.54), which finally reduces to

$$= \frac{1}{(4\pi)^{\frac{n}{2}}} \cdot \frac{1}{4} \sum_{k,s} (D_{x_k}^2 g^{ss}) \,. \tag{4.2.73}$$

Likewise, we find that

$$\sum_{l,k}(D_{x_k}b_l)\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\xi_l\xi_k\,e^{-|\xi|^2}\,d\xi\right)=\frac{1}{(4\pi)^{\frac{n}{2}}}\cdot\frac{1}{2}\sum_k(D_{x_k}b_k)\,. \tag{4.2.74}$$

For the summands above recall that b_k (the first - order term of P) is given by

$$b_k=\sum_l\frac{1}{2}|g|^{-1}g^{kl}(D_{x_l}|g|)+(D_{x_l}\,g^{kl})+ia_k \tag{4.2.75}$$

where $|g|=\det(g)$. Thus

$$\begin{aligned}D_{x_k}b_k=\sum_l\frac{1}{2}|g|^{-1}\Big(&(D_{x_k}g^{kl})(D_{x_l}|g|)+g^{kl}(D^2_{x_k,x_l}|g|)\\&-|g|^{-1}g^{kl}(D_{x_k}|g|)(D_{x_l}|g|)+2|g|(D^2_{x_k,x_l})g^{kl}\Big)+iD_{x_k}a_k\end{aligned} \tag{4.2.76}$$

and the right hand side reduces in normal coordinates to

$$\frac{1}{2}D^2_{x_k}|g|+\sum_l D^2_{x_k,x_l}g^{kl}+iD_{x_k}a_k \tag{4.2.77}$$

where

Proposition 4.2.3. *With the notation above we have (in normal coordinates)*

$$\frac{1}{2}D^2_{x_k}|g|=\frac{1}{2}\sum_l D^2_{x_k}g_{ll}\,. \tag{4.2.78}$$

Proof. Indeed, the last equation is immediate from the expansion of the determinant on terms of the Levi-Civita symbol:

$$\begin{aligned}D^2_{x_k}|g|&=D^2_{x_k}\left(\sum_{i_1,i_2,\ldots i_n=1}^{n}\varepsilon_{i_1\ldots i_n}g_{1i_1}\ldots g_{ni_n}\right)\\&=\sum_{i_1,i_2,\ldots i_n=1}^{n}\varepsilon_{i_1\ldots i_n}D^2_{x_k}(g_{1i_1}\ldots g_{ni_n})\end{aligned} \tag{4.2.79}$$

where

$$D^2_{x_k}(g_{1i_1}\ldots g_{ni_n})=D_{x_k}\left(\sum_{l=1}^{n}(D_{x_k}g_{li_l})\prod_{s\neq l}g_{si_s}\right)$$

$$= \sum_{l=1}^{n} (D_{x_k}^2 g_{li_l}) \prod_{s\neq l} g_{si_s} + \underbrace{\sum_{l=1}^{n} (D_{x_k} g_{li_l})(D_{x_k} \prod_{s\neq l} g_{si_s})}_{=0 \text{ in norm. coordinates}}$$

$$= \sum_{l=1}^{n} (D_{x_k}^2 g_{li_l}) \prod_{s\neq l} g_{si_s} = D_{x_k}^2 g_{ll}\,, \tag{4.2.80}$$

the last equation following from the fact that $g_{ij} = \delta_{ij}$ at the point where the normal coordinates are centered , so $\prod_{s\neq l} g_{si_s} = 0$ whenever it contains terms that are not on the diagonal. Substitution of (4.2.80) into (4.2.79) then gives (4.2.78). $\square$

Thus, coming back to the term (4.2.71) we deduce that

$$\frac{1}{(4\pi)^{\frac{n}{2}}} \cdot \frac{1}{2} \sum_k (D_{x_k} b_k)$$

$$= \frac{1}{(4\pi)^{\frac{n}{2}}} \cdot \frac{1}{2} \sum_k \left(\frac{1}{2} D_{x_k}^2 |g| + \sum_l D_{x_k,x_l}^2 g^{kl} + i D_{x_k} a_k \right) \qquad \text{(substitute (4.2.77))}$$

$$= \frac{1}{(4\pi)^{\frac{n}{2}}} \cdot \frac{1}{2} \sum_k \left(\frac{1}{2} \sum_l D_{x_k}^2 g_{ll} + \sum_l D_{x_k,x_l}^2 g^{kl} + i D_{x_k} a_k \right) \quad \text{(substitute (4.2.78))}$$

$$= \frac{1}{(4\pi)^{\frac{n}{2}}} \cdot \frac{1}{4} \sum_{k,l} \left(D_{x_k}^2 g_{ll} + 2 D_{x_k,x_l}^2 g^{kl} + 2i D_{x_k} a_k \right). \tag{4.2.81}$$

Next we determine the Gaussian integrals in the term (4.2.70). We know that only summands with even powers of the ξ_i contribute, hence

$$\begin{aligned}
&- \frac{2}{3} \sum_{l,p,q,k} (D_{x_k,x_l}^2 g^{pq}) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_l \xi_p \xi_q \xi_k \, e^{-|\xi|^2} \, d\xi \right) \\
&= -\frac{2}{3} \sum_s (D_{x_s}^2 g^{ss}) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_s^4 \, e^{-|\xi|^2} \, d\xi \right) \\
&\quad - \frac{2}{3} \sum_{\substack{s,p \\ s\neq p}} (D_{x_p,x_s}^2 g^{sp}) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_s^2 \xi_p^2 \, e^{-|\xi|^2} \, d\xi \right) \\
&\quad - \frac{2}{3} \sum_{\substack{s,p \\ s\neq p}} (D_{x_p,x_s}^2 g^{ps}) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_s^2 \xi_p^2 \, e^{-|\xi|^2} \, d\xi \right) \\
&\quad - \frac{2}{3} \sum_{\substack{s,p \\ s\neq p}} (D_{x_s}^2 g^{pp}) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi_s^2 \xi_p^2 \, e^{-|\xi|^2} \, d\xi \right).
\end{aligned}$$

By symmetry of g we have $D^2_{x_k,x_l} g^{sp} = D^2_{x_k,x_l} g^{ps}$ so the two middle terms can be summed together. Splitting up the integrals into one - dimensional factors (using Fubini's theorem) we are left with

$$
\begin{aligned}
&-\frac{2}{3}\sum_{s}(D^2_{x_s} g^{ss})\left(\frac{1}{2\pi}\int_{\mathbb{R}} \xi_s^4\, e^{-\xi_s^2}\, d\xi\right)\prod_{l\neq s}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\xi_l^2}\, d\xi \\
&-\frac{4}{3}\sum_{\substack{s,p\\ s\neq p}}(D^2_{x_p,x_s} g^{sp})\left(\frac{1}{2\pi}\int_{\mathbb{R}} \xi_s^2\, e^{-\xi_s^2}\, d\xi\right)\left(\frac{1}{2\pi}\int_{\mathbb{R}} \xi_p^2\, e^{-\xi_p^2}\, d\xi\right)\prod_{l\neq s,p}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\xi_l^2}\, d\xi \\
&-\frac{2}{3}\sum_{\substack{s,p\\ s\neq p}}(D^2_{x_s} g^{pp})\left(\frac{1}{2\pi}\int_{\mathbb{R}} \xi_s^2\, e^{-\xi_s^2}\, d\xi\right)\left(\frac{1}{2\pi}\int_{\mathbb{R}} \xi_p^2\, e^{-\xi_p^2}\, d\xi\right)\prod_{l\neq s,p}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\xi_l^2}\, d\xi\,.
\end{aligned}
$$

Finally from (4.2.57), using well - known values for $\Gamma(z)$, this is equal to

$$
\begin{aligned}
&-\frac{2}{3}\sum_{s}(D^2_{x_s} g^{ss})\left(\frac{1}{2\pi}\sqrt{\pi}\cdot\frac{3}{4}\right)\prod_{l\neq s}\frac{1}{2\pi}\sqrt{\pi} \\
&-\frac{4}{3}\sum_{\substack{s,p\\ s\neq p}}(D^2_{x_p,x_s} g^{sp})\left(\frac{1}{2\pi}\frac{\sqrt{\pi}}{2})\right)^2\prod_{l\neq s,p}\frac{1}{2\pi}\sqrt{\pi} \\
&-\frac{2}{3}\sum_{\substack{s,p\\ s\neq p}}(D^2_{x_s} g^{pp})\left(\frac{1}{2\pi}\frac{\sqrt{\pi}}{2})\right)^2\prod_{l\neq s,p}\frac{1}{2\pi}\sqrt{\pi} \\
&=\frac{1}{(4\pi)^{\frac{n}{2}}}\Bigg(-\frac{1}{2}\sum_{s}(D^2_{x_s} g^{ss})-\frac{1}{3}\sum_{\substack{s,p\\ s\neq p}}(D^2_{x_p,x_s} g^{sp})-\frac{1}{6}\sum_{\substack{s,p\\ s\neq p}}(D^2_{x_s} g^{pp})\Bigg)\,. \qquad (4.2.82)
\end{aligned}
$$

Lastly, by similar reasoning we deduce that

$$
\begin{aligned}
&-\frac{1}{2}\sum_{k,l} a_k a_l\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\xi_k\xi_l\, e^{-|\xi|^2}\, d\xi\right) \\
&=-\frac{1}{2}\sum_{k} a_k^2\left(\frac{1}{(2\pi)}\int_{\mathbb{R}}\xi_k^2\, e^{-\xi_k^2}\, d\xi\right)\prod_{s\neq k}\frac{1}{(2\pi)}\int_{\mathbb{R}} e^{-\xi_s^2}\, d\xi \\
&=-\frac{1}{2}\sum_{k} a_k^2\left(\frac{1}{(2\pi)}\frac{\sqrt{\pi}}{2}\right)\prod_{s\neq k}\frac{1}{(2\pi)}\sqrt{\pi} \\
&=-\frac{1}{(4\pi)^{\frac{n}{2}}}\cdot\frac{1}{4}\sum_{k} a_k^2\,. \qquad (4.2.83)
\end{aligned}
$$

We have now computed each term in the expression of $c_{\frac{-n+2}{2}}(x)$ and it remains to substitute the results into lines (4.2.69)-(4.2.72). With $D_{x_k} = -i\partial_k$ where $\partial_k := \frac{\partial}{\partial x_k}$

this yields

$$c_{\frac{-n+2}{2}}(x) = \frac{1}{(4\pi)^{\frac{n}{2}}}\Bigg(-\frac{1}{4}\sum_{k,s}(\partial^2_{x_k} g^{ss}) + \frac{1}{2}\sum_{s}(\partial^2_{x_s} g^{ss}) + \frac{1}{3}\sum_{\substack{s,p\\ s\neq p}}(\partial^2_{x_p,x_s} g^{sp})$$
$$+ \frac{1}{6}\sum_{\substack{s,p\\ s\neq p}}(\partial^2_{x_s} g^{pp}) - \frac{1}{4}\sum_{k,s}(\partial^2_{x_k} g_{ss}) - \frac{1}{2}\sum_{k,l}(\partial^2_{x_k,x_l} g^{kl})$$
$$+ \frac{1}{2}\sum_{k}\partial_{x_k} a_k - \frac{1}{4}\sum_{k} a_k^2\Bigg) . \tag{4.2.84}$$

From the second - order Taylor series (4.2.64) and (4.2.65) of the metric in normal coordinates we have

$$\partial^2_{pq}\, g^{ij} = -\frac{1}{3}(R_{iqjp} + R_{ipjq}) = -\partial^2_{pq} g_{ij}\,, \tag{4.2.85}$$

so we can write (4.2.84) purely in terms of derivatives of the inverse metric and gather like terms:

$$c_{\frac{-n+2}{2}}(x) = \frac{1}{(4\pi)^{\frac{n}{2}}}\Bigg(-\frac{1}{4}\sum_{k,s}(\partial^2_{x_k} g^{ss}) + \frac{1}{2}\sum_{s}(\partial^2_{x_s} g^{ss}) + \frac{1}{3}\sum_{\substack{s,p\\ s\neq p}}(\partial^2_{x_p,x_s} g^{sp})$$
$$+ \frac{1}{6}\sum_{\substack{s,p\\ s\neq p}}(\partial^2_{x_s} g^{pp}) + \frac{1}{4}\sum_{k,s}(\partial^2_{x_k} g^{ss}) - \frac{1}{2}\sum_{k,l}(\partial^2_{x_k,x_l} g^{kl}) + \frac{1}{2}\sum_{k}\partial_{x_k} a_k - \frac{1}{4}\sum_{k} a_k^2\Bigg)$$
$$= \frac{1}{(4\pi)^{\frac{n}{2}}}\Bigg(\frac{1}{6}\sum_{\substack{s,p\\ s\neq p}}(\partial^2_{x_p} g^{ss}) - \frac{1}{6}\sum_{\substack{s,p\\ s\neq p}}(\partial^2_{x_p,x_s} g^{sp}) + \frac{1}{2}\sum_{k}\partial_{x_k} a_k - \frac{1}{4}\sum_{k} a_k^2\Bigg) .$$

Substituting (4.2.85) into the above equation yields

$$c_{\frac{-n+2}{2}}(x) = \frac{1}{(4\pi)^{\frac{n}{2}}}\Bigg(\frac{1}{3}\Big(-\frac{2}{6}\sum_{\substack{s,p\\ s\neq p}} R_{spsp} + \frac{1}{6}\sum_{\substack{s,p\\ s\neq p}}(R_{sspp} + R_{spps})\Big) + \frac{1}{2}\sum_{k}\partial_{x_k} a_k - \frac{1}{4}\sum_{k} a_k^2\Bigg) .$$

Lastly, we make explicit the term $\sqrt{\det g}$ (which evaluates to 1 at the centered point x in normal coordinates). Using the Bianchi identity $0 = R_{sspp} + R_{spps} + R_{spsp}$, the above then becomes

$$= \frac{1}{(4\pi)^{\frac{n}{2}}}\Bigg(\frac{1}{3}\Big(-\frac{1}{3}\sum_{\substack{s,p\\ s\neq p}} R_{spsp} - \frac{1}{6}\sum_{\substack{s,p\\ s\neq p}} R_{spsp}\Big) + \frac{1}{2}\sum_{k}\partial_{x_k} a_k - \frac{1}{4}\sum_{k} a_k^2\Bigg)\sqrt{\det g}$$

$$= \frac{1}{(4\pi)^{\frac{n}{2}}} \left(\frac{1}{3} \Big(- \sum_{s<p} R_{spsp} \Big) + \frac{1}{2} \sum_k \partial_{x_k} a_k - \frac{1}{4} \sum_k a_k^2 \right) \sqrt{\det g}\,.$$

Thus we conclude

$$\begin{aligned} c_{\frac{-n+2}{2}} &= \int_M c_{\frac{-n+2}{2}}(x)\,|dx| \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}} \left(\frac{1}{3} \int_M K\, d\mu_g + \frac{1}{2} \int_M \operatorname{div}(A)\, d\mu_g - \frac{1}{4} \int_M |A|^2\, d\mu_g \right) \end{aligned} \tag{4.2.86}$$

where $d\mu_g$ denotes the volume form induced by the Riemannian metric (locally given by $d\mu_g = \sqrt{\det g}\, dx$) and

$$\begin{aligned} K &= -\sum_{i<j} R_{ijij} && \text{the scalar curvature,} \\ \operatorname{div}(A) &= \frac{1}{\sqrt{|g|}} \sum_i \partial_i \left(\sqrt{|g|}\, a_i \right) && \text{the Riemannian divergence,} \\ |A|^2 &= \sum_{ij} g_{ij} a_i a_j && \text{the Riemannian length,} \end{aligned}$$

evaluated in normal coordinates.

This finishes our application of resolvent symbol formulae to heat trace coefficients. Let us now turn to another application, namely the direct computation of index formulae.

4.3 Resolvent symbols on Riemann surfaces and the Riemann Roch formula

Here we shall be concerned with another application of resolvent symbols, namely for proving topological identities via heat kernel coefficient calculations. As an illustration we find closed formulas for the first three terms in the resolvent symbol expansion corresponding to a Dirac - Laplacian on a Riemann Surface. These are then applied to recover the well - known Riemann-Roch formula via a direct computation, avoiding Duhamel's principle as well as the Getzler rescaling.

4.3.1 Preliminaries

Let us introduce the basic setting for the Riemann-Roch formula as described in [22]. Let M be a compact boundaryless Riemann surface with smooth positive definite metric. On a coordinate patch $U \subset M$ the latter is given by

$$h(z, \bar{z})\, dz \otimes d\bar{z} \tag{4.3.1}$$

and the induced volume element is written as

$$d\mathrm{Vol} = \frac{i}{2} h(z, \bar{z})\, dz \wedge d\bar{z}. \tag{4.3.2}$$

Let $\mathcal{V} \to M$ be a holomorphic vector bundle of rank n with typical fibre V. It is determined by the transition functions

$$g_{lj} \colon U_j \cap U_l \to \mathrm{GL}(n, \mathbb{C}), \tag{4.3.3}$$

defined over each non-empty intersection of local coordinate neighbourhoods. By use of a partition of unity we define a Hermitian structure for $\mathcal{V}$ via a system $\{E \colon U \to \mathrm{GL}(n, \mathbb{C})\}$ of locally defined positive definite Hermitian matrix - valued maps, varying smoothly over their domain, and satisfying

$$g_{lj}^* \, E_l \, g_{lj} = E_j \quad \text{ on } U_j \cap U_l. \tag{4.3.4}$$

This induces a Hermitian structure on the determinant bundle det $E \to M$ (the line bundle whose typical fibre is the top exterior power $\Lambda^n V$), with transition rule

$$\det\, E_l \; |\det\, g_{lj}|^2 = \det\, E_j \quad \text{ on } U_j \cap U_l. \tag{4.3.5}$$

The complexified cotangent bundle T^*M splits into a direct sum

$$T^*M \cong \Lambda^{1,0}T^*M \oplus \Lambda^{0,1}T^*M \tag{4.3.6}$$

where for each point $p \in M$, dz is a basis for $\Lambda^{1,0}T_p^*M$ and likewise $d\bar{z}$ serves as a basis for $\Lambda^{0,1}T_p^*M$. By patching together the local data we can see that the de

Rham operator d decomposes into $d = \partial + \bar{\partial}$ so that, for example if f is a smooth function whose support is contained in a coordinate neighbourhood U we may write $df = \partial f + \bar{\partial} f = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z}$.

The first Chern class of $\mathcal{V}$ is then represented by the differential form

$$\partial\bar{\partial}\log(\det\, E) \in \Lambda^{1,1}\, T^*M$$

from which we deduce the Chern number

$$c_1(\mathcal{V}) = \frac{1}{2\pi i}\int_M \partial\bar{\partial}\log(\det\, E). \tag{4.3.7}$$

Likewise, the differential form $\partial\bar{\partial}\log h \in \Lambda^{1,1}\, T^*M$ is the canonical representative which is used to compute the Chern number,

$$c_1(M) = \frac{1}{2\pi i}\int_M \partial\bar{\partial}\log h\,. \tag{4.3.8}$$

Next we also need to consider analytical information about the manifold M. Define the differential operator $\triangle\colon \Gamma^\infty(\mathcal{V}) \to \Gamma^\infty(\mathcal{V})$ acting on the space of smooth sections of $\mathcal{V}$, by patching together the local formula

$$\triangle \;=\; -\,(hE^T)^{-1}\frac{\partial}{\partial z}\left(E^T\,\frac{\partial}{\partial \bar{z}}\right)\,. \tag{4.3.9}$$

This operator is elliptic (meaning its leading symbol invertible) and of second order. Furthermore, if we denote by $L^2(\mathcal{V})$ the completion of $\Gamma^\infty(\mathcal{V})$ with respect to the norm induced by the inner product $(u,v) = \int_M (E\bar{u})^T v\, d\mathrm{Vol}$ then the following properties of $\triangle$ are well known:

Proposition 4.3.1. *[22] The operator $\triangle$ has non-negative discrete spectrum $0 \leq \mu_0 \leq \mu_1 \cdots \to \infty$. The corresponding eigensections are smooth and form a complete orthogonal basis for $L^2(\mathcal{V})$. The eigenspaces $E_\triangle(\mu_k) = \{\,\phi \in \Gamma^\infty(\mathcal{V}) \mid \triangle\phi = \mu_k\phi\,\}$ are of finite dimension, moreover*

$$E_\triangle(0) = Ker(\triangle) = H^0(\mathcal{V}) \tag{4.3.10}$$

where the right - hand side denotes the space of holomorphic sections of the vector bundle $\mathcal{V}$.

Further, let us denote by $\widetilde{\mathcal{V}} := \Lambda^{1,0}\, T^*M \otimes \mathcal{V}^*$ the tensor product of the canonical line bundle and the dual bundle. We define similarly an operator $\widetilde{\triangle}\colon \Gamma^\infty(\widetilde{\mathcal{V}}) \to \Gamma^\infty(\widetilde{\mathcal{V}})$ by pasting together the local formula

$$\widetilde{\triangle} = -E\,\frac{\partial}{\partial z}\left((hE)^{-1}\frac{\partial}{\partial \bar{z}}\right). \tag{4.3.11}$$

The following theorem summarises the properties of the operator $\widetilde{\triangle}$ as well as its relationship to $\triangle$:

Proposition 4.3.2. *The operator $\widetilde{\triangle}$ is elliptic with the same properties as listed above for $\triangle$, substituting for* (4.3.10) *the equation*

$$E_{\widetilde{\triangle}}(0) = Ker\,(\widetilde{\triangle}) = H^0(\widetilde{\mathcal{V}})\,. \tag{4.3.12}$$

Furthermore, the positive spectra of $\widetilde{\triangle}$ and $\triangle$ correspond in the sense that $\mu > 0$ is an eigenvalue of $\triangle$ if and only if it is an eigenvalue of $\widetilde{\triangle}$, and in this case we have

$$\dim\left(E_{\triangle}(\mu)\right) = \dim\left(E_{\widetilde{\triangle}}(\mu)\right). \tag{4.3.13}$$

We shall take these results as given since we don't require techniques or concepts from the proofs, more details however may be found in [22].
The relation between the analytical and the topological properties is described by the famous Riemann - Roch formula

$$\dim\, H^0(\mathcal{V}) - \dim\, H^0(\widetilde{\mathcal{V}}) = \frac{1}{2\pi i}\int_M \partial\bar{\partial}\log(\det\, E) + n(1-g_M) \tag{4.3.14}$$

where g_M denotes the genus of M (and we recall that n denotes the rank of $\mathcal{V}$). Furthermore, from the Gauss - Bonnet theorem we can recover that

$$2(1-g_M) = \frac{1}{2\pi i}\int_M \partial\bar{\partial}\log h, \tag{4.3.15}$$

substituting this into the right hand side of (4.3.14) and (4.3.10) respectively (4.3.12) into the left hand side we can see that(4.3.14) can be written in the form

$$\dim\, E_{\triangle}(0) - \dim\, E_{\widetilde{\triangle}}(0) = \frac{1}{2\pi i}\int_M \partial\bar{\partial}\log(\det\, E) + \frac{n}{4\pi i}\int_M \partial\bar{\partial}\log h\,. \tag{4.3.16}$$

Finally let us consider the heat operator $e^{-t\triangle}$ in this context. We described the heat operator in Section 4.2.1, in fact all the properties needed there (such as ellipticity and the discrete spectrum along the non-negative half of the real line) are fulfilled by our present operators so we shall refer to the treatment there rather than repeating it. For each $t > 0$ the associated heat kernel $k_{\triangle}(t, x, y)$ is an element of $\Gamma(\mathcal{V} \hat{\otimes} \mathcal{V}^*)$, the space of smooth sections of the exterior tensor product of the bundles $\mathcal{V}$ and $\mathcal{V}^*$ and can be written in terms of the normalised eigenfunctions of $\triangle$ as

$$k_{\triangle}(t, x, y) = \sum_k e^{-t\mu_k} \phi_k(x) \otimes (E\,\overline{\phi}_k)^T(y) \tag{4.3.17}$$

As a pseudodifferential operator the order of $e^{-t\triangle}$ is arbitrarily low so it has a classical trace given by

$$\mathrm{Tr}(e^{-t\triangle}) = \int_M \mathrm{tr}\big(k_{\triangle}(t, x, x)\big)\ |dx| = \sum_k e^{-t\mu_k} \tag{4.3.18}$$

where $\mathrm{tr}\,(k_{\triangle}(t, x, x)) = \sum_k e^{-t\mu_k} (E\,\overline{\phi}_k)^T(x) \phi_k(x)$ and $|dx|$ identifies locally with Lebesgue measure. Similarly, for $e^{-t\widetilde{\triangle}}$ we have

$$\mathrm{Tr}(e^{-t\widetilde{\triangle}}) = \int_M \mathrm{tr}\big(k_{\widetilde{\triangle}}(t, x, x)\big)\ |dx| = \sum_k e^{-t\widetilde{\mu}_k}. \tag{4.3.19}$$

Now since the positive spectra of $\triangle$ and $\widetilde{\triangle}$ cancel out we can form the difference of (4.3.18) and (4.3.19) and see that for $t > 0$

$$\mathrm{Tr}(e^{-t\triangle}) - \mathrm{Tr}(e^{-t\widetilde{\triangle}}) = \sum_{\mu_k \geq 0} e^{-t\mu_k} - \sum_{\widetilde{\mu}_k \geq 0} e^{-t\widetilde{\mu}_k} = \dim\, E_{\triangle}(0) - \dim\, E_{\widetilde{\triangle}}(0) \tag{4.3.20}$$

where we note that the right hand side is independent of t. On the other hand, recall from Section 4.2.3 that there exist short time asymptotic expansions

$$\mathrm{Tr}(e^{-tP}) \quad \backsim_{t \to 0_+} \quad \sum_{j \geq 0} c_{\frac{-k+j}{m}}\, t^{\frac{-k+j}{m}} \tag{4.3.21}$$

where P denotes one of the operators $\triangle, \widetilde{\triangle}$ and k is the real dimension of the manifold and m denotes the order of the operator P, so in our case we have

$$k = m = 2$$

so let us take on the concrete case from here. The heat coefficients $c_{\frac{-2+j}{2}}$ are given by

$$c_{\frac{-2+j}{2}} = \int_M \operatorname{tr}\left(c_{\frac{-2+j}{2}}(x)\right) |dx| \tag{4.3.22}$$

and the integrand is locally of the form

$$c_{\frac{-2+j}{2}}(x) = \int_{\mathbb{R}^n} \int_\gamma e^{-\lambda} r_{-2-j}(x,\xi,\lambda)\, đ\lambda đ\xi \tag{4.3.23}$$

with $đ\lambda = id\lambda/(2\pi)$ and $đ\xi = (2\pi)^{-n} d\xi$ (c.f. equations (4.2.44) and (4.2.45)). In the next section we determine, as in the case before of the Laplace - Beltrami operator, concrete formulae for the resolvent symbols in order to calculate the integrals.

4.3.2 Explicit formulae for the resolvent symbols

First we decompose the local symbol of the operator $\triangle$ into homogeneous parts,

$$\sigma_\triangle(x,\xi) = a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi) \tag{4.3.24}$$

where $a_k(x,t\xi) = t^k a_k(x,\xi)$. In view of the identities

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} + \frac{1}{i}\frac{\partial}{\partial x_2}) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} - \frac{1}{i}\frac{\partial}{\partial x_2})$$

we expand (4.3.9) in terms of differentiation by real local coordinates x_1, x_2, this yields

$$\triangle = -g(\partial_{x_1}^2 + \partial_{x_2}^2) - \alpha(i\partial_{x_1} - \partial_{x_2}) \tag{4.3.25}$$

with $\partial_{x_i} = \frac{\partial}{\partial x_i}$ for $i \in \{1,2\}$ and

$$g = \frac{h^{-1}}{4}\,, \tag{4.3.26}$$

$$\alpha = \frac{1}{2i}(hE^T)^{-1}\frac{\partial E^T}{\partial z}\,. \tag{4.3.27}$$

Then replacing $-i\partial_{x_i}$ with ξ_i we obtain the corresponding symbol

$$\sigma_\triangle = g|\xi|^2 + \alpha(\xi_1 + i\xi_2) \tag{4.3.28}$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2$. (More precisely, $g|\xi|^2 = g|\xi|^2 I_n$ where I_n denotes the $n \times n$ identity matrix. We shall adopt the common abuse of notation $kI_n = k$ whenever k is a scalar.) Thus in homogeneous terms, (4.3.28) decomposes into

$$a_2 = g|\xi|^2 \qquad a_1 = \alpha(\xi_1 + i\xi_2) \qquad a_0 = 0\,. \tag{4.3.29}$$

In a similar manner we obtain the symbol for $\widetilde{\triangle}$:

$$\sigma_{\widetilde{\triangle}} = g|\xi|^2 + \widetilde{\alpha}(\xi_1 + i\xi_2) \tag{4.3.30}$$

where g is as above and

$$\widetilde{\alpha} = \frac{1}{2i} E \frac{\partial (hE)^{-1}}{\partial z}\,. \tag{4.3.31}$$

The homogeneous components are therefore

$$\widetilde{a}_2 = g|\xi|^2 \qquad \tilde{a}_1 = \tilde{\alpha}(\xi_1 + i\xi_2) \qquad \tilde{a}_0 = 0. \tag{4.3.32}$$

We can now state and prove the main theorem in this section.

Theorem 4.3.3. *With the notation above, let $r(x, \xi, \lambda)$ denote the local symbol of the resolvent operator $(\triangle - \lambda)^{-1}$. Then the first three resolvent symbols in the asymptotic series $r(x, \xi, \lambda) \backsim \sum_{j \geq 0} r_{-2-j}(x, \xi, \lambda)$ are given by*

$$r_{-2} = (g|\xi|^2 - \lambda)^{-1} \tag{4.3.33}$$

$$r_{-3} = 2r_{-2}^3 \sum_l g(D_{x_l} g)\, \xi_l |\xi|^2 \; - r_{-2}^2 \alpha(\xi_1 + i\xi_2) \tag{4.3.34}$$

and

$$\begin{aligned} r_{-4} = {} & 12 r_{-2}^5 \sum_{k,l} g^2 (D_{x_k} g)(D_{x_l} g)\, \xi_k \xi_l |\xi|^4 - 2r_{-2}^4 \sum_k g\,(D_{x_k} g)^2\, |\xi|^4 \\ & - 4r_{-2}^4 \sum_{k,l} g (D_{x_k} g)(D_{x_l} g)\, \xi_k \xi_l |\xi|^2 - 4r_{-2}^4 \sum_{k,l} g^2 (D^2_{x_k, x_l} g)\, \xi_k \xi_l |\xi|^2 \\ & - 6 r_{-2}^4 \sum_k \alpha\, g (D_{x_k} g)(\xi_1 + i\xi_2) \xi_k |\xi|^2 \\ & + r_{-2}^3 \sum_k g (D^2_{x_k} g)\, |\xi|^2 + 2r_{-2}^3 \sum_k (D_{x_k} \alpha) g (\xi_1 + i\xi_2) \xi_k \\ & + r_{-2}^3\, \alpha (D_{x_1} g) |\xi|^2 + i r_{-2}^3 \alpha (D_{x_2} g) |\xi|^2 + r_{-2}^3 \alpha^2 (\xi_1 + i\xi_2)^2\,. \end{aligned} \tag{4.3.35}$$

Proof. Recall from (4.2.9) and (4.2.10) the recursive definition of the resolvent symbols:

$$r_{-2} = (a_2 - \lambda)^{-1} \tag{4.3.36}$$

$$r_{-2-j} = -r_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!}(\partial_\xi^\mu a_{2-k}) \cdot (D_x^\mu r_{-2-l}) \tag{4.3.37}$$

where in this case $\mu = (\mu_1, \mu_2)$ denotes a multi - index, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $x = (x_1, x_2) \in \mathbb{R}^2$, furthermore $D_x^\mu = (-i\partial_x)^\mu$ with $\partial_{x_i} = \partial/\partial x_i$ whilst $\partial_x^\mu = \partial_{x_1}^{\mu_1}\partial_{x_2}^{\mu_2}$. Now the term r_{-2} This is given directly by (4.3.36). Next for r_{-3} the calculation is similar to the analogous term in the previous section: we see from (4.3.37) that

$$r_{-3} = -r_{-2} \sum_{\substack{|\mu|+k+l=1 \\ l<1}} \frac{1}{\mu!}\partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l}\,. \tag{4.3.38}$$

We have $l = 0$ throughout; furthermore $|\mu| + k + l = 1$ means that the factorial term simplifies to $\mu! = 1$ in all summands. Thus

$$r_{-3} = -r_{-2} \sum_{|\mu|+k=1} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2} \tag{4.3.39}$$

which splits into the summands

$$\underbrace{-r_{-2}\left(\sum_l \partial_{\xi_l} a_2 D_{x_l} r_{-2}\right)}_{|\mu|=1,k=0} \underbrace{-r_{-2}\left(a_1 r_{-2}\right)}_{|\mu|=0,k=1}\,. \tag{4.3.40}$$

For the first term we note that $\partial_{\xi_l} a_2 = 2g\,\xi_l$; moreover

$$D_{x_l} r_{-2}^m = -\,m\,r_{-2}^{m+1}\,(D_{x_l} g)|\xi|^2 \tag{4.3.41}$$

for any positive integer m. Substituting this as well as $a_1 = \alpha(\xi_1 + i\xi_2)$ gives

$$\begin{aligned} r_{-3} &= \underbrace{-r_{-2}\left(\sum_l (2g\xi_l)\left(-r_{-2}^2\,(D_{x_l} g)|\xi|^2\right)\right)}_{|\mu|=1,k=0} \underbrace{-r_{-2}\left(\alpha(\xi_1 + i\xi_2) r_{-2}\right)}_{|\mu|=0,k=1} \\ &= 2r_{-2}^3 \sum_l g(D_{x_l} g)\,\xi_l|\xi|^2 \; - r_{-2}^2\alpha(\xi_1 + i\xi_2) \end{aligned} \tag{4.3.42}$$

and the last expression is precisely the right hand side of (4.3.34).

Let us then proceed to

$$r_{-4} = -r_{-2} \sum_{\substack{|\mu|+k+l=2 \\ l<2}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l} \,. \tag{4.3.43}$$

With $l < 2$ there are two terms corresponding to $l = 0$ or $l = 1$:

$$r_{-4} = \underbrace{-r_{-2} \sum_{|\mu|+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2}}_{l=0} \quad \underbrace{-r_{-2} \sum_{|\mu|+k=1} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-3}}_{l=1} \tag{4.3.44}$$

which in turn split into summands according to $k = 0, 1$ or 2 in the first and $k = 0$ or 1 in the second expression. We now look more closely at these:

Term corresponding to $l = 0$: We have

$$\begin{aligned} & - r_{-2} \sum_{|\mu|+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2} \\ &= \underbrace{-r_{-2} \sum_{|\mu|=2} \frac{1}{\mu!} \partial_\xi^\mu a_2 D_x^\mu r_{-2}}_{k=0} \quad \underbrace{-r_{-2} \sum_{|\mu|=1} \frac{1}{\mu!} \partial_\xi^\mu a_1 D_x^\mu r_{-2}}_{k=1} \end{aligned} \tag{4.3.45}$$

(The last summand related to $k = 2$ vanishes because $a_0 = 0$). Note that

$$\partial_{\xi_k} \left(\partial_{\xi_l} a_2 \right) = \partial_{\xi_k} \left(2g\xi_l \right) = 2g \, \delta_{kl} \tag{4.3.46}$$

where δ_{kl} is the Kronecker delta, and

$$\begin{aligned} D_{x_l} \left(D_{x_k} r_{-2} \right) &= D_{x_l} \left(-r_{-2}^2 \left(D_{x_k} g \right) |\xi|^2 \right) \\ &= 2r_{-2}^3 \left(D_{x_l} g \right) \left(D_{x_k} g \right) |\xi|^4 - r_{-2}^2 \left(D_{x_l,x_k}^2 g \right) |\xi|^2 \,. \end{aligned} \tag{4.3.47}$$

Application of the above as well as (4.3.46) shows that the first term of (4.3.45) is given by

$$-r_{-2} \sum_{|\mu|=2} \frac{1}{\mu!} \partial_\xi^\mu a_2 D_x^\mu r_{-2} = -r_{-2} \sum_k \frac{1}{2} (2g) D_{x_k}^2 r_{-2}$$

$$= -2r_{-2}^4 \sum_k g\,(D_{x_k} g)^2\,|\xi|^4 + r_{-2}^3 \sum_k g(D_{x_k}^2 g)\,|\xi|^2. \quad (4.3.48)$$

For the second term in (4.3.45) we note that

$$\partial_{\xi_l} a_1 = \partial_{\xi_l}\Big(\alpha(\xi_1 + i\xi_2)\Big) = \begin{cases} \alpha & \text{if } l = 1 \\ i\alpha & \text{if } l = 2 \end{cases} \quad (4.3.49)$$

and

$$D_{x_l} r_{-2} = -r_{-2}^2\,(D_{x_l} g)|\xi|^2\,, \quad (4.3.50)$$

therefore

$$\begin{aligned} -r_{-2} \sum_{|\mu|=1} \frac{1}{\mu!} \partial_\xi^\mu a_1 D_x^\mu r_{-2} &= -r_{-2} \sum_l \partial_{\xi_l} a_1 D_{x_l} r_{-2} \\ &= \quad r_{-2}^3\,\alpha(D_{x_1} g)|\xi|^2 + i r_{-2}^3 \alpha(D_{x_2} g)|\xi|^2 \quad (4.3.51) \end{aligned}$$

and we can now fully expand (4.3.45) into

$$\begin{aligned} &\underbrace{-2r_{-2}^4 \sum_k g\,(D_{x_k} g)^2\,|\xi|^4 + r_{-2}^3 \sum_k g(D_{x_k}^2 g)\,|\xi|^2}_{\text{term } (4.3.48)} \\ &\qquad + \quad \underbrace{r_{-2}^3\,\alpha(D_{x_1} g)|\xi|^2 + i r_{-2}^3 \alpha(D_{x_2} g)|\xi|^2}_{\text{term } (4.3.51)}\,. \end{aligned} \quad (4.3.52)$$

Term corresponding to $l = 1$: Repeating the procedure here we start with

$$\begin{aligned} &- r_{-2} \sum_{|\mu|+k=1} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-3} \\ &= -r_{-2} \underbrace{\sum_{|\mu|=1} \frac{1}{\mu!} \partial_\xi^\mu a_2 D_x^\mu r_{-3}}_{k=0} \quad - \quad r_{-2} \underbrace{\sum_{|\mu|=0} \frac{1}{\mu!} \partial_\xi^\mu a_1 D_x^\mu r_{-3}}_{k=1} \\ &= -r_{-2} \sum_k \partial_{\xi_k} a_2 \underbrace{D_{x_k} r_{-3}}_{(*)} \quad - \quad r_{-2}(a_1 r_{-3})\,. \end{aligned} \quad (4.3.53)$$

All the terms involved are known by now except for $(*)$ which we determine below:

$$D_{x_k} r_{-3} \overset{(4.3.42)}{=} D_{x_k}\Big(2r_{-2}^3 \sum_l g(D_{x_l} g)\,\xi_l|\xi|^2 \; - r_{-2}^2 \alpha(\xi_1 + i\xi_2)\Big)$$

and a direct computation shows this is

$$= -6r_{-2}^4 \sum_l g(D_{x_k}g)(D_{x_l}g)\,\xi_l|\xi|^4 + 2r_{-2}^3 \sum_l (D_{x_k}g)(D_{x_l}g)\,\xi_l|\xi|^2$$
$$+ 2r_{-2}^3 \sum_l g(D^2_{x_k,x_l}g)\,\xi_l|\xi|^2 + 2r_{-2}^3\,\alpha(D_{x_k}g)(\xi_1+i\xi_2)|\xi|^2 - r_{-2}^2(D_{x_k}\alpha)(\xi_1+i\xi_2)\,.$$

In conclusion (4.3.53) therefore expands into

$$\begin{aligned}
&-2r_{-2}\sum_k g\Bigg(-6r_{-2}^4\sum_l g(D_{x_k}g)(D_{x_l}g)\,\xi_l|\xi|^4 + 2r_{-2}^3\sum_l (D_{x_k}g)(D_{x_l}g)\,\xi_l|\xi|^2 \\
&\quad + 2r_{-2}^3\sum_l g(D^2_{x_k,x_l}g)\,\xi_l|\xi|^2 + 2r_{-2}^3\,\alpha(D_{x_k}g)(\xi_1+i\xi_2)|\xi|^2 \\
&\quad - r_{-2}^2(D_{x_k}\alpha)(\xi_1+i\xi_2)\Bigg)\xi_k - r_{-2}\,\alpha(\xi_1+i\xi_2)\,\left(2r_{-2}^3\sum_l g(D_{x_l}g)\,\xi_l|\xi|^2\; - r_{-2}^2\alpha(\xi_1+i\xi_2)\right) \\
&= 12r_{-2}^5\sum_{k,l} g^2(D_{x_k}g)(D_{x_l}g)\,\xi_k\xi_l|\xi|^4 - 4r_{-2}^4\sum_{k,l} g(D_{x_k}g)(D_{x_l}g)\,\xi_k\xi_l|\xi|^2 \\
&\quad - 4r_{-2}^4\sum_{k,l} g^2(D^2_{x_k,x_l}g)\,\xi_k\xi_l|\xi|^2 - 4r_{-2}^4\,\sum_k \alpha\,g(D_{x_k}g)(\xi_1+i\xi_2)\xi_k|\xi|^2 \\
&\quad + 2r_{-2}^3\sum_k (D_{x_k}\alpha)g(\xi_1+i\xi_2)\xi_k - 2r_{-2}^4\sum_l \alpha\,g(D_{x_l}g)\,(\xi_1+i\xi_2)\xi_l|\xi|^2\; + r_{-2}^3\alpha^2(\xi_1+i\xi_2)^2\,.
\end{aligned} \tag{4.3.54}$$

Finally we substitute (4.3.52) and (4.3.54) into the recursive formula for r_{-4}. Rearranging and collecting like terms, we obtain

$$\begin{aligned}
r_{-4} &= \underbrace{-r_{-2}\sum_{|\mu|+k=2}\frac{1}{\mu!}\partial_\xi^\mu a_{2-k}D_x^\mu r_{-2}}_{\text{equation (4.3.52)}}\quad \underbrace{-r_{-2}\sum_{|\mu|+k=1}\frac{1}{\mu!}\partial_\xi^\mu a_{2-k}D_x^\mu r_{-3}}_{\text{equation (4.3.54)}} \\
&= 12r_{-2}^5\sum_{k,l} g^2(D_{x_k}g)(D_{x_l}g)\,\xi_k\xi_l|\xi|^4 - 2r_{-2}^4\sum_k g\,(D_{x_k}g)^2\,|\xi|^4 \\
&\quad - 4r_{-2}^4\sum_{k,l} g(D_{x_k}g)(D_{x_l}g)\,\xi_k\xi_l|\xi|^2 - 4r_{-2}^4\sum_{k,l} g^2(D^2_{x_k,x_l}g)\,\xi_k\xi_l|\xi|^2 \\
&\quad - 6r_{-2}^4\,\sum_k \alpha\,g(D_{x_k}g)(\xi_1+i\xi_2)\xi_k|\xi|^2 \\
&\quad + r_{-2}^3\sum_k g(D^2_{x_k}g)\,|\xi|^2 + 2r_{-2}^3\sum_k (D_{x_k}\alpha)g(\xi_1+i\xi_2)\xi_k
\end{aligned}$$

$$+ r_{-2}^3\,\alpha(D_{x_1}g)|\xi|^2 + ir_{-2}^3\alpha(D_{x_2}g)|\xi|^2 + r_{-2}^3\alpha^2(\xi_1+i\xi_2)^2\,, \tag{4.3.55}$$

as required. □

Note that the homogeneous terms a_k and $\widetilde{a}_k$ (listed in (4.3.29) respectively (4.3.32)) differ only in the coefficients α and $\widetilde{\alpha}$, so replacing α by $\widetilde{\alpha}$ in the above formulas immediately gives the first three terms for the asymptotic series of the local resolvent symbol corresponding to the operator $(\widetilde{\triangle}-\lambda)^{-1}$. This will be important in the next section where we derive the Riemann-Roch formula.

Corollary 4.3.4. *Let $\widetilde{r}(x,\xi,\lambda)$ denote the local symbol of the resolvent operator $(\widetilde{\triangle}-\lambda)^{-1}$, then the first three resolvent symbols in the asymptotic series $\widetilde{r}(x,\xi,\lambda) \backsim \sum_{j\geq 0}\widetilde{r}_{-2-j}(x,\xi,\lambda)$ are given by*

$$\widetilde{r}_{-2} = (g|\xi|^2-\lambda)^{-1} \tag{4.3.56}$$

$$\widetilde{r}_{-3} = 2\widetilde{r}_{-2}^3\sum_l g(D_{x_l}g)\,\xi_l|\xi|^2 \;-\widetilde{r}_{-2}^2\widetilde{\alpha}(\xi_1+i\xi_2) \tag{4.3.57}$$

and

$$\begin{aligned}
\widetilde{r}_{-4} = {} & 12\widetilde{r}_{-2}^5\sum_{k,l} g^2(D_{x_k}g)(D_{x_l}g)\,\xi_k\xi_l|\xi|^4 - 2\widetilde{r}_{-2}^4\sum_k g\,(D_{x_k}g)^2\,|\xi|^4 \\
& - 4\widetilde{r}_{-2}^4\sum_{k,l} g(D_{x_k}g)(D_{x_l}g)\,\xi_k\xi_l|\xi|^2 - 4\widetilde{r}_{-2}^4\sum_{k,l} g^2(D^2_{x_k,x_l}g)\,\xi_k\xi_l|\xi|^2 \\
& - 6\widetilde{r}_{-2}^4\sum_k \widetilde{\alpha}\,g(D_{x_k}g)(\xi_1+i\xi_2)\xi_k|\xi|^2 \\
& + \widetilde{r}_{-2}^3\sum_k g(D^2_{x_k}g)\,|\xi|^2 + 2\widetilde{r}_{-2}^3\sum_k (D_{x_k}\widetilde{\alpha})g(\xi_1+i\xi_2)\xi_k \\
& + \widetilde{r}_{-2}^3\,\widetilde{\alpha}(D_{x_1}g)|\xi|^2 + i\widetilde{r}_{-2}^3\widetilde{\alpha}(D_{x_2}g)|\xi|^2 + \widetilde{r}_{-2}^3\widetilde{\alpha}^2(\xi_1+i\xi_2)^2.
\end{aligned} \tag{4.3.58}$$

4.3.3 The Riemann-Roch formula

We can now come back to our original motivation for computing the resolvent symbols above, which is to derive the Riemann Roch formula written in the form

(4.3.16). Recall (c.f. (4.3.20)) that the entry point for the resolvent symbols comes from the short time heat trace expansion via the identity

$$\operatorname{Tr}(e^{-t\triangle}) - \operatorname{Tr}(e^{-t\widetilde{\triangle}}) = \dim\, E_{\triangle}(0) - \dim\, E_{\widetilde{\triangle}}(0) \qquad (t > 0)\,.$$

For small $t > 0$ we may replace the left hand side by the respective expansions

$$\sum_{j\geq 0} c_{\frac{-2+j}{2}}\, t^{\frac{-2+j}{2}} - \sum_{j\geq 0} \widetilde{c}_{\frac{-2+j}{2}}\, t^{\frac{-2+j}{2}} = \dim\, E_{\triangle}(0) - \dim\, E_{\widetilde{\triangle}}(0)\,,$$

and since the right hand side does not depend on t one can deduce that $c_{\frac{-2+j}{2}} - \widetilde{c}_{\frac{-2+j}{2}} = 0$ whenever $j \neq 2$ whilst

$$c_0 - \widetilde{c}_0 = \int_M \operatorname{tr}\,(c_0 - \widetilde{c}_0)(x)\ |dx| = \dim\, E_{\triangle}(0) - \dim\, E_{\widetilde{\triangle}}(0)\,. \tag{4.3.59}$$

Hence, if we can show that

$$\int_M \operatorname{tr}\,(c_0 - \widetilde{c}_0)(x)\ |dx| = \frac{1}{2\pi i}\int_M \partial\bar{\partial}\log(\det\, E) + \frac{n}{4\pi i}\int_M \partial\bar{\partial}\log h \tag{4.3.60}$$

then we arrive at the Riemann Roch formula.

Before we start the calculations let us note some general properties that we shall need. First we recall the identity (4.2.48) here for convenience (adjusted to the current setting):

$$\frac{i}{2\pi}\int_\gamma e^{-\lambda}(g|\xi|^2 - \lambda)^{-k} d\lambda = \frac{1}{(k-1)!} e^{-g|\xi|^2}. \tag{4.3.61}$$

We also require (4.2.57) again, this time with a non-trivial positive parameter $\beta > 0$ (which already appeared in (4.2.54)). Since we are in the special case of dimension 2 let us recall the here only for that situation:

$$\int_{\mathbb{R}^2} \xi_1^{2n_1}\xi_2^{2n_2} e^{-\beta|\xi|^2}\, d\xi = \frac{\Gamma(n_1+\frac{1}{2})\,\Gamma(n_2+\frac{1}{2})}{\beta^{(n_1+\frac{1}{2})+(n_2+\frac{1}{2})}} \qquad (\beta > 0) \tag{4.3.62}$$

where n_1, n_2 denote positive integers. The powers in the polynomial term in (4.3.62) are even; for the odd case we have

$$\int_{\mathbb{R}^2} \xi_1^{n_1}\xi_2^{n_2}\, e^{-\beta|\xi|^2}\, d\xi = 0 \qquad \text{(at least one of } n_1, n_2 \text{ odd).} \tag{4.3.63}$$

Finally we list some basic properties of the trace and determinant of finite - dimensional matrices. For any invertible $n \times n$ matrices P, Q depending on a parameter t and for any scalar λ one has

$$\begin{aligned}
\mathrm{tr}(P+\lambda Q) &= \mathrm{tr}(P) + \lambda \mathrm{tr}(Q) \\
\mathrm{tr}(P^T) &= \mathrm{tr}(P) \\
\det(P^T) &= \det(P) \\
\mathrm{tr}(PQ) &= \mathrm{tr}(QP) \\
\partial_t \log\det P &= \mathrm{tr}(P^{-1}\partial_t P) \\
\mathrm{tr}\,(\partial_t P) &= \partial_t \mathrm{tr}(P)\,.
\end{aligned} \tag{4.3.64}$$

where $\partial_t = \partial/\partial t$.

Using these observations let us next determine the heat coefficients (in fact for the Riemann Roch formula we only need h_0).

The heat coefficient c_{-1} This calculation here is relatively short using the integrals above, indeed from (4.3.23) and (4.3.33) one has

$$c_{-1}(x) = \int_{\mathbb{R}^2} \left(\frac{i}{2\pi} \int_\gamma e^{-\lambda} (g|\xi|^2 - \lambda)^{-1}\, d\lambda \right) đ\xi$$

and the inner integral is evaluated by (4.3.61) so that the right hand side is

$$= \int_{\mathbb{R}^2} e^{-g|\xi|^2} đ\xi \tag{4.3.65}$$

which in turn simplifies (by (4.3.62)) to

$$= (4\pi^2 g)^{-1}\, \Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) = (4\pi\, g)^{-1}\,. \tag{4.3.66}$$

Now from (4.3.26) we see that $(4g)^{-1} = h$. Also, under the identification $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ we have $\frac{i}{2} dz \wedge d\bar{z} = dx_1 dx_2$. Thus, locally

$$\begin{aligned}
\mathrm{tr}(c_{-1}(x))\, dx_1 dx_2 &= \mathrm{tr}((4\pi\, g)^{-1} I_n)\, dx_1 dx_2 \\
&= \frac{\mathrm{tr}(I_n)}{\pi} h\, \frac{i}{2} dz \wedge d\bar{z} = \frac{n}{\pi}\, d\mathrm{Vol},
\end{aligned}$$

therefore

$$c_{-1} = \int_M \operatorname{tr}\left(c_{-1}(x)\right)\,|dx| = \frac{n}{\pi}\int_M d\mathrm{Vol} = \frac{n}{\pi}\,\mathrm{Vol}(M). \tag{4.3.67}$$

The heat coefficient $c_{-\frac{1}{2}}$ The next heat coefficient $c_{-\frac{1}{2}}$ evaluates to zero since all summands of

$$r_{-3} = 2r_{-2}^3 \sum_l g(D_{x_l}g)\,\xi_l|\xi|^2 \; - r_{-2}^2\alpha(\xi_1 + i\xi_2)$$

involve odd powers of ξ_i, so the evaluation

$$c_{-\frac{1}{2}}(x) = \int_{\mathbb{R}^2}\int_\gamma e^{-\lambda}\, r_{-3}(x,\xi,\lambda)\,đ\lambda đ\xi = 0 \tag{4.3.68}$$

follows from (4.3.63) and hence

$$c_{-\frac{1}{2}} = \int_M \operatorname{tr}\left(c_{-\frac{1}{2}}(x)\right)|dx| = 0\,. \tag{4.3.69}$$

The heat coefficient c_0 This is the third heat coefficient (the constant term in the expansion). We need to evaluate

$$c_0(x) = \int_{\mathbb{R}^2}\int_\gamma e^{-\lambda}\, r_{-4}(x,\xi,\lambda)\,đ\lambda đ\xi\,. \tag{4.3.70}$$

The full expression for r_{-4} is given in (4.3.35), but for the computation we only need to take into account the terms that contain no odd powers of ξ (odd powers integrate to zero by (4.3.63)). Thus substituting the relevant terms for r_{-4} gives

$$\begin{aligned}
c_0(x) = \int_{\mathbb{R}^2}\int_\gamma e^{-\lambda}\Bigg(& 12r_{-2}^5 \sum_k g^2(D_{x_k}g)^2\,\xi_k^2|\xi|^4 - 2r_{-2}^4 \sum_k g\,(D_{x_k}g)^2\,|\xi|^4 \\
& - 4r_{-2}^4 \sum_k g(D_{x_k}g)^2\,\xi_k^2|\xi|^2 - 4r_{-2}^4 \sum_k g^2(D_{x_k}^2 g)\,\xi_k^2|\xi|^2 \\
& + r_{-2}^3 \sum_k g(D_{x_k}^2 g)\,|\xi|^2 \\
& - 6r_{-2}^4\,\alpha\, g(D_{x_1}g)\xi_1^2|\xi|^2 - 6ir_{-2}^4\,\alpha\, g(D_{x_2}g)\xi_2^2|\xi|^2 \\
& + 2r_{-2}^3(D_{x_1}\alpha)g\xi_1^2 + 2ir_{-2}^3(D_{x_2}\alpha)g\xi_2^2 \\
& + r_{-2}^3\,\alpha(D_{x_1}g)|\xi|^2 + ir_{-2}^3\alpha(D_{x_2}g)|\xi|^2 + r_{-2}^3\alpha^2(\xi_1^2 - \xi_2^2)\Bigg)\,đ\lambda đ\xi\,.
\end{aligned}$$

Using (4.3.61) to evaluate the contour integral and obtain

$$
\begin{aligned}
c_0(x) = \int_{\mathbb{R}^2} e^{-g|\xi|^2} \Bigg(& \frac{1}{2} \sum_k g^2 (D_{x_k} g)^2 \, \xi_k^2 |\xi|^4 - \frac{1}{3} \sum_k g \, (D_{x_k} g)^2 \, |\xi|^4 \\
& - \frac{2}{3} \sum_k g (D_{x_k} g)^2 \, \xi_k^2 |\xi|^2 - \frac{2}{3} \sum_k g^2 (D_{x_k}^2 g) \, \xi_k^2 |\xi|^2 \\
& + \frac{1}{2} \sum_k g (D_{x_k}^2 g) \, |\xi|^2 \\
& - \alpha \, g (D_{x_1} g) \xi_1^2 |\xi|^2 - i \, \alpha \, g (D_{x_2} g) \xi_2^2 |\xi|^2 \\
& + (D_{x_1} \alpha) g \xi_1^2 + i \, (D_{x_2} \alpha) g \xi_2^2 \\
& + \frac{1}{2} \alpha (D_{x_1} g) |\xi|^2 + \frac{1}{2} i \alpha (D_{x_2} g) |\xi|^2 + \frac{1}{2} \alpha^2 (\xi_1^2 - \xi_2^2) \Bigg) đ\xi \, . \qquad (4.3.71)
\end{aligned}
$$

Next we apply (4.3.62) and (4.2.56) in order compute the Gaussian integrals and collect like terms; this yields

$$
\begin{aligned}
c_0(x) = \frac{1}{4\pi} \Bigg(& - \frac{1}{6} D_{x_1} \left(g^{-1} (D_{x_1} g) \right) - \frac{1}{6} D_{x_2} \left(g^{-1} (D_{x_2} g) \right) \\
& + \frac{1}{2} D_{x_1} \left(g^{-1} \alpha \right) + \frac{1}{2} i D_{x_2} \left(g^{-1} \alpha \right) \Bigg) . \qquad (4.3.72)
\end{aligned}
$$

Under the identification $z = x_1 + i x_2$, $\bar{z} = x_1 - i x_2$, we have $D_{x_1} = -i \partial_{x_1} = -i(\partial/\partial_z + \partial/\partial_{\bar{z}}) =: -i(\partial_z + \partial_{\bar{z}})$ and similarly $D_{x_2} = \partial_z - \partial_{\bar{z}}$. Substituting these as well as $g = (4h)^{-1}$ and $\alpha = \frac{1}{2i} (h E^T)^{-1} (\partial E^T / \partial z)$ and rearranging the result gives

$$
\begin{aligned}
c_0(x) &= \frac{1}{4\pi} \Bigg(- \frac{2}{3} \partial_z \left(h^{-1} \partial_{\bar{z}} h \right) - 4 i \partial_{\bar{z}} \left(h \alpha \right) \Bigg) \qquad (4.3.73) \\
&= - \frac{1}{6\pi} \partial_z \partial_{\bar{z}} \log h \, I_n - \frac{1}{2\pi} \partial_{\bar{z}} \left((E^T)^{-1} \frac{\partial E^T}{\partial z} \right) . \qquad (4.3.74)
\end{aligned}
$$

Finally, we take the trace and apply the properties listed in (4.3.64) together with the fact that $\partial_z \partial_{\bar{z}} = \partial_{\bar{z}} \partial_z$ to simplify the result:

$$
\begin{aligned}
\operatorname{tr}(c_0(x)) &= - \frac{1}{6\pi} \operatorname{tr} \left(\partial_z \partial_{\bar{z}} \log h \, I_n \right) - \frac{1}{2\pi} \operatorname{tr} \left(\partial_{\bar{z}} \big((E^T)^{-1} \frac{\partial E^T}{\partial z} \big) \right) \\
&= - \frac{n}{6\pi} \partial_z \partial_{\bar{z}} \log h - \frac{1}{2\pi} \partial_z \partial_{\bar{z}} \log \det E \, . \qquad (4.3.75)
\end{aligned}
$$

Hence, locally with $dx_1 dx_2 = \frac{i}{2} dz \wedge d\bar{z}$ this gives

$$
\operatorname{tr}(c_0(x)) \, dx_1 dx_2 = \frac{n}{12\pi i} \partial_z \partial_{\bar{z}} \log h \, dz \wedge d\bar{z} + \frac{1}{4\pi i} \partial_z \partial_{\bar{z}} \log \det E \, dz \wedge d\bar{z}
$$

and therefore

$$c_0 = \int_M \mathrm{tr}\left(c_0(x)\right)|dx| \qquad = \frac{1}{4\pi i}\int_M \partial\bar{\partial}\log\det E + \frac{n}{12\pi i}\int_M \partial\bar{\partial}\log h\,. \tag{4.3.76}$$

4.3.4 Derivation of the Riemann Roch formula

Of course the computations above can also be carried out for the heat coefficients $\widetilde{c}_{\frac{-2+j}{2}}$ that constitute the asymptotic expansion

$$\mathrm{Tr}\left(e^{-t\widetilde{\triangle}}\right) \backsim \sum_{j\geq 0}\widetilde{c}_{\frac{-2+j}{2}}t^{\frac{-2+j}{2}}\,, \tag{4.3.77}$$

however, instead of doing so one can make use of the fact that the local symbols (4.3.28) and (4.3.30) for the operators $\triangle$ and $\widetilde{\triangle}$ differ only in the first order coefficient. This means that the polynomials r_{-2-j}, $\widetilde{r}_{-2-j}$ are identical except in terms involving α, $\widetilde{\alpha}$ respectively. Thus we can for instance deduce immediately that

$$\widetilde{c}_{-1} = \frac{n}{\pi}\,\mathrm{Vol}(M) \tag{4.3.78}$$

because $r_{-2} = \widetilde{r}_{-2}$. The second heat coefficient associated wit $\widetilde{\triangle}$ vanishes

$$\widetilde{c}_{-\frac{1}{2}} = 0\,, \tag{4.3.79}$$

indeed this follows from (4.3.63) together with the observation that $\widetilde{r}_{-3}$ consists only of odd monomials in ξ. For the third heat coefficient $\widetilde{c}_0$ (the constant term in the heat trace expansion) we may simply replace α by $\widetilde{\alpha}$ in (4.3.73) and proceed from there. Thus, recalling that h is scalar and therefore commutes, we compute

$$\begin{aligned}\widetilde{c}_0(x) &= \frac{1}{4\pi}\left(-\frac{2}{3}\partial_z\left(h^{-1}\partial_{\bar{z}}h\right) - 4i\partial_{\bar{z}}\left(h\widetilde{\alpha}\right)\right)\\ &= \frac{1}{3\pi}\partial_z\partial_{\bar{z}}\log h\, I_n + \frac{1}{2\pi}\partial_{\bar{z}}\left(\frac{\partial E}{\partial z}E^{-1}\right)\,.\end{aligned} \tag{4.3.80}$$

Again we take the trace and use the properties (4.3.64) as well as the equation $\partial_z\partial_{\bar{z}} = \partial_{\bar{z}}\partial_z$ to get

$$\mathrm{tr}(\widetilde{c}_0(x)) = \frac{n}{3\pi}\partial_z\partial_{\bar{z}}\log h + \frac{1}{2\pi}\partial_z\partial_{\bar{z}}\log\det E \tag{4.3.81}$$

and with $dx_1dx_2 = \frac{i}{2}dz \wedge d\bar{z}$ this then gives the local formula

$$\operatorname{tr}(\widetilde{c}_0(x))|dx| = -\frac{n}{6\pi i}\partial_z\partial_{\bar{z}} \log h \, dz \wedge d\bar{z} - \frac{1}{4\pi i}\partial z\partial_{\bar{z}} \log \det E \, dz \wedge d\bar{z} \,. \qquad (4.3.82)$$

Thus

$$\widetilde{c}_0 = \int_M \operatorname{tr}(\widetilde{c}_0(x)) \, |dx| = -\frac{1}{4\pi i}\int_M \partial\bar{\partial} \log \det E - \frac{n}{6\pi i}\int_M \partial\bar{\partial} \log h \,. \qquad (4.3.83)$$

The Riemann Roch formula Finally, recall from (4.3.60) that the Riemann Roch Formula is equivalent to

$$\int_M \operatorname{tr}(c_0 - \widetilde{c}_0)(x) \, |dx| = \frac{1}{2\pi i}\int_M \partial\bar{\partial} \log(\det \, E) + \frac{n}{4\pi i}\int_M \partial\bar{\partial} \log h \,. \qquad (4.3.84)$$

This equation now follows immediately from our computations for the heat coefficients, namely we calculated

$$c_0 = \int_M \operatorname{tr}(c_0(x)) \, |dx| = \frac{1}{4\pi i}\int_M \partial\bar{\partial} \log \det E + \frac{n}{12\pi i}\int_M \partial\bar{\partial} \log h$$

and

$$\widetilde{c}_0 = \int_M \operatorname{tr}(\widetilde{c}_0(x)) \, |dx| = -\frac{1}{4\pi i}\int_M \partial\bar{\partial} \log \det E - \frac{n}{6\pi i}\int_M \partial\bar{\partial} \log h \,.$$

By subtracting the second from the first equation we arrive at (4.3.84).

Chapter 5

On log-polyhomogeneous symbols and the canonical trace over a simple warped product

5.1 Introduction

In this final chapter we report on work in progress related to a current research project. Let $\pi\colon E \to M$ be a smooth vector bundle over an n-dimensional compact Riemannian manifold M without boundary and consider a classical pseudodifferential operator (ψdo) $A\colon C^\infty(M;E) \to C^\infty(M;E)$ with symbol σ. Provided A has non-integer order, the *canonical trace* $\mathrm{TR}(A)$, first introduced by Kontsevich and Vishik [25], is defined by the formula

$$\mathrm{TR}(A) := \int_M \mathrm{Tr}_x(A)\, dx \tag{5.1.1}$$

where dx identifies locally with Lebesgue measure and

$$\mathrm{TR}_x(A) := -\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}_x\big(\sigma(x,\xi)\big) đ\xi \tag{5.1.2}$$

is a finite - part integral obtained from the local asymptotic expansion of σ (the finite part integral is defined as the (unique) constant term in an asymptotic expansion of a divergent integral). The canonical trace extends the classical trace, which is well defined on smoothing operators, to pseudodifferential operators whose order is not contained in $\mathbb{Z} \cap [-\dim M, \infty)$.

In this chapter we study an extension of this canonical trace to pseudodifferential operators with suitable symbols defined over a *simple warped product.* The latter is a product manifold $\mathcal{M} := [0, \infty) \times M$, where M is a compact manifold without boundary, endowed with a metric of the form $dr^2 + h^2(r)g$ where g is a metric on M and $h\colon [0, \infty) \to \mathbb{R}$ is a smooth positive function. For the current study we shall restrict our attention to those cases where $h \to \infty$ as $r \to \infty$. Examples are metrics where $h(r) = r^k$ (k a positive integer, the case $k = 2$ is the metric cone), or hyperbolic space, where $h(r) = \cosh(r)$.

The symbols we shall admit are *log - polyhomogeneous* as defined by Lesch in [27] and moreover exhibit the log - polyhomogeneous property in the "radial" space variable that parametrises the factor $[0, \infty)$. Defining such symbol classes and proving they are closed under the usual symbol product will be the topic of Section 5.2. Once we have established a symbol calculus we can consider possible extensions of the canonical trace. To this end in Section 5.3 and 5.4 we study finite - part integrals (the standard technique used in the context of the canonical trace) to give meaning to the expression

$$\int_{\mathcal{M}} \int_{T_p^*\mathcal{M}} \sigma(p, \theta)\, đ\theta dr dx \tag{5.1.3}$$

which is divergent in general due to the non-compact factor $[0, \infty)$ in $\mathcal{M}$. Here σ is an element of the symbol class we consider, $T_p^*\mathcal{M}$ denotes the cotangent space at a point p in $\mathcal{M}$, and $đ\theta = (2\pi)^{-n-1} d\theta$ is normalised Lebesgue measure. One way to proceed here is to apply a further finite-part integral and set

$$\int_{\mathcal{M}} \int_{T_p^*\mathcal{M}} \sigma(p, \theta)\, đ\theta dr dx := ⨍_{[0,\infty)} \mathrm{TR}_r(\mathsf{Op}[\sigma])\, dr \tag{5.1.4}$$

where $\mathsf{Op}[\sigma]$ is the pseudodifferential operator acting on a function u by $\mathsf{Op}[\sigma](u) = \mathcal{F}^{-1}[\sigma(\theta)\hat{u}(\theta)]$ (here $\mathcal{F}^{-1}$ denotes inverse Fourier transformation whilst $f \mapsto \hat{f}$

stands for Fourier transformation and variables not involved in this process are omitted), and

$$\mathrm{TR}_r(A) = \int_M \mathrm{TR}_{(r,x)}(A)\, dx \tag{5.1.5}$$

is a parametrised family of canonical traces analogous to (5.1.1) and (5.1.2). In fact, making this work is what motivates the assumption of an additional log-polyhomogeneous expansion of our symbols in the radial direction. In Section 5.4.1 we find that (5.1.5) defines a global density under certain circumstances that are different but analogous to the corresponding situation on closed manifolds (as described for example in [37, Proposition 1.10]), in particular the condition of non-integer order remains sufficient whilst the condition of even - even symbols applies to even - dimensional manifolds M whereas the condition of even - odd symbols is sufficient for odd - dimension M (this is reversed in the standard setting). In such cases one can express the right hand side of (5.1.4) in terms of integrals over the factors involving strongly polyhomogeneous symbols as defined by G. Grubb and R. Seeley [16], this is shown in Theorem 5.4.5.

Finally in Section 5.5 we turn our attention to the study of an example which is of particular interest to us, namely the resolvent and complex powers of the Laplace - Beltrami operator on a warped product. For the moment we concentrate here on the symbol expansion; an analysis of the corresponding canonical trace will follow.

5.2 Symbols of log-polyhomogeneous growth on $[0,\infty) \times M$

Let us first recall basic definitions and relevant properties of classical and log - polyhomogeneous symbol classes on open subsets of Euclidean space, for a more detailed exposition of the standard theory see [41, 27].

Let $U \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$, let $\mu \in \mathbb{C}$, let V be a finite dimensional normed vector space. A smooth function $a(x,\xi) \in C^\infty\big(U \times \mathbb{R}^n, \mathrm{End}(V)\big) \cong$

$C^\infty\big(T^*U, \mathrm{End}(V)\big)$ is an element of the symbol class $S^\mu(U,V)$ if for any compact subset $K \subset U$ and any multi - indices α, β there exists a constant $C_{\alpha\beta K}$ so that for all $x \in K$ and $\xi \in \mathbb{R}^n$

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha\beta K}(1+|\xi|)^{\mathrm{Re}(\mu)-|\beta|} \tag{5.2.1}$$

with respect to a choice of norm $|\cdot|$ on $\mathrm{End}(V)$. The parameter $\mu \in \mathbb{C}$ is called the *order* of a. We denote

$$S(U,V) := \bigcup_{\mu\in\mathbb{C}} S^\mu(U,V) \quad \text{and} \quad S^{-\infty}(U,V) := \bigcap_{r\in\mathbb{R}} S^r(U,V)\,.$$

A *classical symbol of order* μ is a symbol $a(x,\xi) \in S^\mu(U,V)$ for which there exists an asymptotic expansion

$$a(x,\xi) \backsim_{\xi\to\infty} \sum_{j\geq 0} a_{\mu-j}(x,\xi)$$

where each term $a_{\mu-j}$ is a symbol of order $\mu - j$ and homogeneous in ξ for $|\xi| \geq 1$ of order $\mu - j$, that is $a_{\mu-j}(x,t\xi) = t^{\mu-j}(x,\xi)$ for $t \geq 1$ and $|\xi| \geq 1$.

Remark 5.2.1. The meaning of $\backsim_{\xi\to\infty}$ is that for ξ large and any N,

$$a(x,\xi) - \sum_{j=0}^{N-1} a_{\mu-j}(x,\xi) \quad \in \quad S^{\mu-N}(U,V)\,.$$

Equivalently, given any positive integer N there exist functions $a_{\mu-j}(x,\xi) \in S^{\mu-j}(U,V)$, $0 \leq j \leq N$ which are ξ - homogeneous of degree $\mu - j$ (as described above), and a symbol $a_N \in S^{\mu-N}(U,V)$ such that

$$a(x,\xi) = \sum_{j=0}^{N-1} a_{\mu-j}(x,\xi) + a_N(x,\xi)\,. \tag{5.2.2}$$

The set of classical symbols of order μ is denoted by $\mathrm{CS}^\mu(U,V)$, furthermore we set

$$\mathrm{CS}(U,V) := \bigcup_{\mu\in\mathbb{C}} \mathrm{CS}^\mu(U,V)\,.$$

We also need the larger class of log - polyhomogeneous symbols introduced by Lesch [27]. For these we choose a C^∞ function $[\cdot]\colon \mathbb{R}^n \to (0,\infty)$ that agrees with the usual Euclidean norm outside the unit ball, that is $[\mathbf{y}] = |\mathbf{y}|$ for $\mathbf{y} \notin \overline{B}_1(\mathbf{0}) = \{\mathbf{y} \in \mathbb{R}^n \colon |\mathbf{y}| \leq 1\}$. Keeping all the previous notation, fix in addition a non - negative integer k. A *log - polyhomogeneous symbol of order $\mu \in \mathbb{C}$ and* log *degree* k is a function $a(x,\xi) \in C^\infty\big(T^*U, \mathrm{End}(V)\big)$ for which there exists an asymptotic expansion

$$a(x,\xi) \backsim_{\xi\to\infty} \sum_{j\geq 0} a_{\mu-j}(x,\xi) \tag{5.2.3}$$

in the sense that for large ξ, any positive integer N and any $\varepsilon > 0$,

$$a(x,\xi) - \sum_{j=0}^{N-1} a_{\mu-j}(x,\xi)$$

is an element of $S^{\mu-N+\varepsilon}(U,V)$. Furthermore, each term in the expansion on the right of (5.2.3) is assumed to be representable in the form

$$a_{\mu-j}(x,\xi) = \sum_{i=0}^{k} a_{\mu-j,i}(x,\xi)\log^i[\xi] \tag{5.2.4}$$

with $a_{\mu-j,i}(x,\xi)$ homogeneous in ξ of degree $\mu - j$ as described above. Equivalently, this means that for any positive integer N we can write $a(x,\xi)$ in the form

$$\begin{aligned} a_{\mu-j}(x,\xi) &= \sum_{j=0}^{N-1} \underbrace{a_{\mu-j}(x,\xi)}_{\text{in } C^\infty(T^*U,\,\mathrm{End}(V))} + \underbrace{a_N(x,\xi)}_{\substack{\text{in } S^{\mu-N+\varepsilon} \\ \text{any } \varepsilon>0}} & (5.2.5)\\ &= \sum_{j=0}^{N-1}\sum_{i=0}^{k} \underbrace{a_{\mu-j,i}(x,\xi)}_{\substack{\xi\text{-homog. deg.} \\ \mu-j \text{ for } |\xi|\geq 1}} \log^i[\xi] + \underbrace{a_N(x,\xi)}_{\substack{\text{in } S^{\mu-N+\varepsilon} \\ \text{any } \varepsilon>0}}\,. & (5.2.6)\end{aligned}$$

Provided $a(x,\xi)$ satisfies the above condition we write

$$a(x,\xi) \backsim \sum_{j=0}^{\infty} a_{\mu-j}(x,\xi) = \sum_{j=0}^{\infty}\sum_{i=0}^{k} a_{\mu-j,i}(x,\xi)\log^i[\xi]\,.$$

We denote by $\mathrm{LS}^{\mu,k}(U,V)$ the set of log - polyhomogeneous symbols of order μ and log - degree k and set

$$\mathrm{LS}(U,V) := \bigcup_{k\in\mathbb{N}_0} \mathrm{LS}^{*,k}(U,V) \quad \text{where} \quad \mathrm{LS}^{*,k}(U,V) := \bigcup_{\mu\in\mathbb{C}} \mathrm{LS}^{\mu,k}(U,V)$$

Remark 5.2.2. The class of log - polyhomogeneous symbols incorporates the class of classical symbols since $\mathrm{CS}(U,V) = \mathrm{LS}^{*,0}(U,V)$.

Let us now proceed to the class of symbols that is adapted to our case of interest. We are dealing with a non - compact, cylindrical manifold and the non-compactness introduces new problems as far as traces of pseudodifferential operators are concerned, since their definition involves integration over the manifold. The idea is to require that the local symbols and their derivatives grow log - polyhomogeneously in the "radial" direction, a condition that is already present in the cotangent variable ξ. Then one can adapt regularisation techniques used to deal with divergences of ξ - integrals over T_x^*U to regularise the integral along the radial direction.

First we set up a notion of "log-polyhomogeneous radial growth" similar to the kind of growth exhibited by Lesch's symbols in the cotangent variables. Fix a smooth positive function $f\colon [0,\infty) \to \mathbb{R}$. For illustrative purposes we shall in later sections describe a concrete example with $f(r) = r$ for $r \geq 1/2$. It is important to keep in mind that the function f fixed here is not necessarily related to the warping function that appears in the metric of a warped product.

Definition 5.2.3. Let $\mathcal{O} = [0,\infty) \times U$ with U an open subset in $\mathbb{R}^n$. We denote a point in $\mathcal{O}$ by (r,x) where $x \in U$. A *log - polyhomogeneous symbol of (double) order* $(\nu,\mu) \in \mathbb{C}\times\mathbb{C}$ *and* log - *degree* (k_1,k_2) is a member of the class $\mathrm{LS}^{\mu,k_1}(\mathcal{O},V)$ such that for each function $a_{\mu-j,i}(r,x,\xi)$ in the asymptotic expansion

$$a(r,x,\xi) \backsim_{\xi\to\infty} \sum_{j=0}^{\infty}\sum_{i=0}^{k_1} a_{\mu-j,i}(r,x,\xi)\log^i[\xi] \tag{5.2.7}$$

(here ξ refers to the components of a vector cotangent to $[0,\infty)\times U$ at the point (r,x)) there exists an additional asymptotic expansion

$$a_{\mu-j,i}(r,x,\xi) \backsim_{r\to\infty} \sum_{s\geq 0} a_{\nu-s,\mu-j,i}(r,x,\xi)\,, \tag{5.2.8}$$

where, as before in the ξ-direction, the summands on the right hand side of (5.2.8) are assumed smooth and to have a representation of the form

$$a_{\nu-s,\mu-j,i}(r,x,\xi) = \sum_{l=0}^{k_2} a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r) \tag{5.2.9}$$

where in addition to the usual ξ-homogeneity

$$a_{\nu-s,l,\mu-j,i}(f(r),x,\alpha\xi) = \alpha^{\mu-j} a_{\nu-s,l,\mu-j,i}(f(r),x,\xi) \quad \text{for } \alpha \geq 1, |\xi| \geq 1$$

the coefficients exhibit homogeneity in $f(r)$ of decreasing degree, concretely

$$a_{\nu-s,l,\mu-j,i}(f(r),x,\xi) = f(r)^{\nu-s} a_{\nu-s,l,\mu-j,i}(1,x,\xi) \quad \text{for } r \geq 1\,. \tag{5.2.10}$$

Provided $a(r,x,\xi)$ satisfies the above condition we write

$$a(r,x,\xi) \backsim \sum_{j=0}^{\infty}\sum_{i=0}^{k_1}\sum_{s=0}^{\infty}\sum_{l=0}^{k_2} a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r)\log^i[\xi] \tag{5.2.11}$$

and denote by $\mathrm{L_2S}^{\nu,\mu,k_1,k_2}(\mathcal{O},V)$ the set of log - polyhomogeneous symbols of order $(\nu,\mu) \in \mathbb{C}\times\mathbb{C}$ and log - degree (k_1,k_2). Finally, we also set

$$\mathrm{L_2S}(\mathcal{O},V) := \bigcup_{k_1\in\mathbb{N}_0}\bigcup_{k_2\in\mathbb{N}_0} \mathrm{L_2S}^{*,*,k_1,k_2}(\mathcal{O},V) \tag{5.2.12}$$

where

$$\mathrm{L_2S}^{*,*,k_1,k_2}(\mathcal{O},V) := \bigcup_{\nu\in\mathbb{C}}\bigcup_{\mu\in\mathbb{C}} \mathrm{L_2S}^{\,\nu,\mu,k_1,k_2}(\mathcal{O},V)\,. \tag{5.2.13}$$

With respect to composition and differentiation the following algebraic properties are satisfied by this class:

Proposition 5.2.4. *If $a \in \mathrm{L_2S}^{\nu,\mu,k_1,k_2}(\mathcal{O},V)$ and $b \in \mathrm{L_2S}^{\nu',\mu',k_1',k_2'}(\mathcal{O},V)$, then for any multi - index $\alpha = (\alpha_0,\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^{n+1}$ we have*

1. *$\partial^\alpha_{(t,x)} a \in \mathrm{L_2S}^{\nu-\alpha_0,\mu,k_1,k_2}(\mathcal{O},V)$ where $t = f(r)$,*
2. *$\partial^\alpha_\xi a \in \mathrm{L_2S}^{\nu,\mu-|\alpha|,k_1,k_2}(\mathcal{O},V)$*
3. *$a\cdot b \in \mathrm{L_2S}^{\nu+\nu',\mu+\mu',k_1+k_1',k_2+k_2'}(\mathcal{O},V)\,.$*

Proof. For (1) and (2) we conduct the differentiation on an asymptotic expansion and show that the resulting expansion has the desired properties. Split $\partial^\alpha_{(t,x)}$ into $\partial_t^{\alpha_0}$ and the remaining differentiation with respect to x, that is $\partial^\alpha_{(t,x)} = \partial_t^{\alpha_0}\partial_x^{\alpha'}$ where $\partial_x^{\alpha'} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$. Then

$$\partial^\alpha_{(t,x)} a \backsim \sum_{j=0}^{\infty}\sum_{i=0}^{k_1}\sum_{s=0}^{\infty}\sum_{l=0}^{k_2} \underbrace{\partial_t^{\alpha_0}\left(\partial_x^{\alpha'} a_{\nu-s,l,\mu-j,i}(t,x,\xi)\log^l t\right)}_{(*)\ \xi\text{-homogeneous of degree } \mu-j}\log^i[\xi]\,. \tag{5.2.14}$$

Furthermore the basic formula

$$\partial_t^{\alpha_0}\Big(g(t)\log^l t\Big) = \sum_{p=0}^{\alpha_0}\binom{\alpha}{p}\partial_t^{\alpha_0-p}g(t)\cdot\partial_t^p\log^l t \tag{5.2.15}$$

with

$$\partial_t^p \log^l t = t^{-p}\sum_{k=0}^{l-1} d_{pk}\log^k t \tag{5.2.16}$$

where the d_{pk} are constants (possibly $=0$) implies that the homogeneity in t is as claimed, since for a function $g(t)$ that is homogeneous in t of order $\nu - s$ we then have

$$\partial_t^{\alpha_0}\big(g(t)\log^l t\big) = \sum_{k=0}^{l-1}\underbrace{\left(\sum_{p=0}^{\alpha_0} d_{pk}\binom{\alpha_0}{p}t^{-p}\partial_t^{\alpha_0-p}g(t)\right)}_{t\text{-homog. degree } \nu-s-\alpha_0}\log^k t\,. \tag{5.2.17}$$

Thus the expression $(*)$ in (5.2.14) is a polynomial in $\log t$ of degree l whose coefficients are t-homogeneous of degree $\nu - s - \alpha_0$. After collecting terms corresponding to the factor $\log^l t$ for $0 \le l \le k_2$ we therefore obtain an asymptotic expansion for $\partial^\alpha_{(t,x)} a$ of the required form.

Next we consider

$$\begin{aligned}\partial_\xi^\alpha a(r,x,\xi) &\backsim \sum_{j=0}^{\infty}\sum_{i=0}^{k_1}\sum_{s=0}^{\infty}\sum_{l=0}^{k_2}\partial_\xi^\alpha\Big(a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r)\log^i[\xi]\Big)\\ &= \sum_{j=0}^{\infty}\sum_{l=0}^{k_1}\sum_{s=0}^{\infty}\sum_{i=0}^{k_2}\Big(\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\underbrace{\partial_\xi^\beta a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)}_{(**)}\log^l f(r)\,\partial_\xi^{\alpha-\beta}\log^i[\xi]\Big)\,.\end{aligned} \tag{5.2.18}$$

As in the case above, the log - degree k_1 will not change. Furthermore $(**)$ is ξ-homogeneous of order $\mu - j - |\beta|$ for $|\xi| \geq 1$ whilst

$$\begin{aligned}\partial_\xi^{\alpha-\beta} \log^i[\xi] &\overset{(*)}{=} \partial_\xi^{\alpha-\beta} \log^i |\xi| && ((*) \text{ for } |\xi| \geq 1) \\ &= |\xi|^{|\beta|-|\alpha|} \sum_{k=i-|\alpha-\beta|}^{i-1} d_k \log^k |\xi| && (\text{some constants } d_k)\end{aligned}$$

contributes a factor that is ξ- homogeneous of order $|\beta| - |\alpha|$ for $|\xi| \geq 1$, so overall the ξ-homogeneity of each summand above is $\mu - j - |\alpha|$. After collecting terms corresponding to the factor $\log^i[\xi]$ for $1 \leq i \leq k_1$ we obtain an asymptotic expansion for $\partial_\xi^\alpha a$ of the required form.

Finally, for the composition property (3) we recall from [41, Theorem 3.4] the formula

$$(a \cdot b)(r, x, \xi) \backsim_{\xi \to \infty} \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha a(r, x, \xi) D_{(r,x)}^\alpha b(r, x, \xi) \tag{5.2.19}$$

where $D_{(r,x)}^\alpha = (-i)^{|\alpha|} \partial_{(r,x)}^\alpha$. Now if a is an element in $\mathrm{L_2S}^{\nu,\mu,k_1,k_2}(\mathcal{O}, V)$ and b belongs to $\mathrm{L_2S}^{\nu',\mu',k_1',k_2'}(\mathcal{O}, V)$ then in particular a and b are log - polyhomogeneous symbols in ξ of order μ and log - degree k_1 respectively μ' and log - degree k_1'. For such symbols we take from [27] that the product $(a \cdot b)(r, x, \xi)$ is a log - polyhomogeneous symbol (in ξ) of order $\mu + \mu'$ and log -degree $k_1 + k_1'$. This remains true here as well since the log - polyhomogeneity in ξ is independent from the additional log - polyhomogeneity in $f(r)$. So it only remains to show that the log - polyhomogeneity in $f(r)$ is satisfied. This can be seen from substituting the asymptotic expansions

$$\partial_\xi^\alpha a(r, x, \xi) \backsim \sum_{j=0}^\infty \sum_{i=0}^{k_1} \sum_{s=0}^\infty \sum_{l=0}^{k_2} u_{\nu-s,l,\mu-|\alpha|-j,i}(f(r), x, \xi) \log^l f(r) \log^i[\xi] \tag{5.2.20}$$

respectively (again with $t = f(r)$)

$$D_{(t,x)}^\alpha b(r, x, \xi) \backsim \sum_{j'=0}^\infty \sum_{i'=0}^{k_1'} \sum_{s'=0}^\infty \sum_{l'=0}^{k_2'} v_{\nu'-\alpha_0-s',l',\mu'-j',i'}(f(r), x, \xi) \log^{l'} f(r) \log^{i'}[\xi] \tag{5.2.21}$$

into (5.2.19). First we obtain

$$\sum_{\alpha}\frac{1}{\alpha!}\partial_\xi^\alpha a(r,x,\xi)D^\alpha_{(r,x)}b(r,x,\xi)\backsim$$

$$\sum_{\alpha}\frac{1}{\alpha!}\left(\sum_{j=0}^{\infty}\sum_{i=0}^{k_1}\sum_{s=0}^{\infty}\sum_{l=0}^{k_2}u_{\nu-s,l,\mu-|\alpha|-j,i}(f(r),x,\xi)\log^l f(r)\log^i[\xi]\right)$$
$$\times\left(\sum_{j'=0}^{\infty}\sum_{i'=0}^{k'_1}\sum_{s'=0}^{\infty}\sum_{l'=0}^{k'_2}v_{\nu'-\alpha_0-s',l',\mu'-j',i'}(f(r),x,\xi)\log^{l'} f(r)\log^{i'}[\xi]\right)$$

Now if we expand partial sums and collect terms according to homogeneity and log - degree this reduces to

$$=\sum_{|\alpha|+j+j'=0}^{\infty}\sum_{i+i'=0}^{k_1+k'_1}\sum_{\alpha_1+s+s'=0}^{\infty}\sum_{l+l'=0}^{k_2+k'_2}\frac{1}{\alpha!}u_{\nu-s,l,\mu-|\alpha|-j,i}(f(r),x,\xi)$$
$$\times\, v_{\nu'-\alpha_1-s',l',\mu'-j',i'}(f(r),x,\xi)\log^{l+l'} f(r)\log^{i+i'}[\xi] \tag{5.2.22}$$

where

$$\frac{1}{\alpha!}u_{\nu-s,l,\mu-|\alpha|-j,i}(f(r),x,\xi)\times v_{\nu'-\alpha_1-s',l',\mu'-j',i'}(f(r),x,\xi) \tag{5.2.23}$$

is ξ-homogeneous of order $\mu+\mu'-(|\alpha|+j+j')$ and $f(r)$-homogeneous of order $\nu+\nu'-(\alpha_1+s+s')$, i.e. the above is an asymptotic expansion of the required form. □

Let us finish this section by pointing out that symbols of the class $\mathrm{L}_2\mathrm{S}^{\nu,\mu,k_1,k_2}(\mathcal{O},V)$ are not invariant under a change of variable $(\tau,y)=F(r,x)$. To see this we recall the following result (which establishes the coordinate invariance of the leading term associated with standard symbols):

Theorem 5.2.5 ([41], Theorem 4.2)**.** *Let U be an open subset of $\mathbb{R}^n$ and consider the pseudodifferential operator A given by*

$$Au(x)=\int_U\int_{\mathbb{R}^n}e^{(x-y)\cdot\xi}a(x,y,\xi)u(y)\,dy đ\xi$$

with amplitude $a(x,y,\xi)\in S^{\mu}(U\times U,V)$ *and symbol*

$$\sigma(x,\xi)\backsim_{\xi\to\infty}\sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}D_{y}^{\alpha}a(x,y,\xi)|_{y=x}\,.$$

If $F\colon U\to F(U)\subset\mathbb{R}^n$ *is a diffeomorphism and* $\widetilde{A}$ *is the pseudodifferential operator defined by*

$$\widetilde{A}f=\big(A(f\circ F)\big)\circ F^{-1}$$

then the symbol $\widetilde{\sigma}$ *of* $\widetilde{A}$ *has the asymptotic expansion*

$$\widetilde{\sigma}(y,\eta)|_{y=F(x)}\backsim_{\xi\to\infty}\sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}\sigma(x,F_{*}^{T}(x)\eta)\cdot D_{z}^{\alpha}e^{iG_{F}(z)\cdot\eta}|_{z=x}\tag{5.2.24}$$

where F_{*} *denotes the derivative of* F *and*

$$G(z)=F(z)-F(x)-F_{*}(x)(z-x)\,.$$

If we were to apply this result to a symbol from the class $\mathrm{L}_2\mathrm{S}^{\nu,\mu,k_1,k_2}(\mathcal{O},V)$ with a diffeomorphism $(r,x)\mapsto F(r,x)$ then, due to the derivative terms that are created by $D^{\alpha}_{(\lambda,z)}e^{iG_F(\lambda,z)\cdot\eta}|_{(\lambda,z)=(r,x)}$, the formula (5.2.24) produces an asymptotic expansion that does not have the required form (5.2.11).

However the operators we associate to our symbols later on are parametrised families of pseudodifferential operators over the factor M so we shall consider diffeomorphisms of the form $x\mapsto F(x)$ where the parameter r is left unchanged. In this case the additional logarithmic scaling in the asymptotic expansion is left intact and the change of variable result in [27, Proposition 3.5] carries over to our setting.

5.3 Finite-part integrals of symbols

In view of traces for operators with symbols from the class $\mathrm{L}_2\mathrm{S}(\mathcal{O},V)$ there are two divergence problems one has to address. Let $a(r,x,\xi)\in\mathrm{L}_2\mathrm{S}^{\nu,\mu,k_1,k_2}(\mathcal{O},V)$. First, locally for a fixed point $(r,x)\in\mathcal{O}=[0,\infty)\times U$, with $U\subset\mathbb{R}^n$, the integral

$$\int_{T^{*}_{(r,x)}\mathcal{O}}a(r,x,\xi)đ\xi\tag{5.3.1}$$

is not necessarily convergent. This issue is of course already present when dealing with log - polyhomogeneous symbols $a(x,\xi) \in \mathrm{LS}^{\mu,k}(U,V)$ over closed manifolds. The usual approach to overcome the problem in this case is to regularise (5.3.1) via a finite - part integral

$$\mathrm{LS}^{\mu,k}(U,V) \ni a(x,\xi) \quad \longmapsto \quad ⨍_{T_x^*U} a(x,\xi)\, đ\xi := K(x) \tag{5.3.2}$$

where $K(x)$ is the constant term in the asymptotic expansion

$$\int_{B_x^*(0,R)} a(x,\xi)\, đ\xi \qquad \backsim_{R\to\infty} \; K(x) + (\text{terms that diverge as } R \to \infty)\,. \tag{5.3.3}$$

The fact that the above expansion exists for such symbols was shown by Lesch in [27] from which we shall recall the essential points of the derivation in the next section. This regularisation procedure works likewise for the integral (5.3.1) because the symbol $a(r,x,\xi)$ is log - polyhomogeneous with respect to ξ.

Secondly, because of the non-compact factor $[0,\infty)$ there is a divergence question when we integrate over the underlying manifold; that is we need to make sense of

$$\int_{[0,\infty)\times M} \left(⨍_{T_x^*U} a(r,x,\xi) đ\xi \right) dx dr\,.$$

Since the growth behaviour of $a(r,x,\xi)$ in the r-direction is similar to the growth behaviour in the cotangent variable ξ it is natural to adopt an approach analogous to (5.3.2) and take a finite-part integral.

5.3.1 Preliminaries

Suppose $a(x,\xi) \in \mathrm{LS}^{\mu,k}(U,V)$ is a log - polyhomogeneous symbol defined on an open subset $U \subset \mathbb{R}^n$. The integral $\int_{T_x^*U} a(x,\xi) đ\xi$ diverges if $\mathrm{Re}(\mu) \geq -n$, but one way to extract a number nevertheless is to define a finite - part integral $⨍$, based on the following Lemma.

Lemma 5.3.1 (Asymptotic expansion for a ξ-integral of a log-polyhomogeneous symbol [27]). *For any*

$$a(x,\xi) \backsim \sum_{j=0}^{\infty} a_{\mu-j}(x,\xi) = \sum_{j=0}^{\infty}\sum_{i=0}^{k} a_{\mu-j,i}(x,\xi)\log^i[\xi]$$

in $\mathrm{LS}^{\mu,k}(U,V)$ *there is an asymptotic expansion as* $R\to\infty$

$$\int_{B_x^*(0,R)} a(x,\xi)đ\xi \backsim K(x) + \sum_{\substack{j=0\\ \mu-j\neq -n}}^{\infty} F_{\mu-j}\big(\log R\big)R^{n+\mu-j} + \Pi\big(a_{-n},\log R\big) \qquad (5.3.4)$$

where $K(x)$ *depends on* x *alone and the remaining terms are described as follows. The coefficients of the sum are given by*

$$F_{\mu-j}\big(\log R\big) = \sum_{i=0}^{k} P_i(a_{\mu-j,i})(\log R)$$

where each P_i *is a polynomial of degree* i *whose coefficients* c_m *depend on* $a_{\mu-j,i}$ *(concretely* $P_i(a_{\mu-j,i})(X) = \sum_{m=0}^{i} c_m(a_{\mu-j,i})X^m$*). The final term in the expansion is given by*

$$\Pi\big(a_{-n},\log R\big) = \sum_{i=0}^{k}\frac{1}{i+1}\left(\int_{S_x^*U} a_{-n,i}(x,\xi)đ_S\xi\right)\log^{i+1} R$$

where S_x^*U *is the unit sphere in* T_x^*U*.*

The constant term in (5.3.4) is defined to be the value of the finite - part integral of $a(x,\xi)$, that is

Definition 5.3.2. The finite - part integral at $x\in U$ associates to a log-polyhomogeneous symbol the constant term in the asymptotic expansion (5.3.4),

$$\mathrm{LS}^{\mu,k}(U,V) \ni a(x,\xi) \quad \longmapsto \quad ⨍_{T_x^*U} a(x,\xi)đ\xi := K(x)\,. \qquad (5.3.5)$$

There is a formula for the finite part integral as described in the next lemma.

Lemma 5.3.3. *For $a(x,\xi) \in \mathrm{LS}^{\mu,k}(U,V)$ and any $N > \mathrm{Re}(\mu) + n$ one has*

$$\begin{aligned}
\fint_{T_x^*U} a(x,\xi)đ\xi &= \sum_{j=0}^{N} \int_{B_x^*(0,1)} a_{\mu-j}(x,\xi)đ\xi + \int_{T_x^*U} a_N(x,\xi)đ\xi \\
&\quad + \sum_{\substack{j=0 \\ \mu-j\neq -n}}^{N} \sum_{i=0}^{k} \frac{(-1)^{i+1} i!}{(\mu-j+n)^{i+1}} \int_{S_x^*U} a_{\mu-j,i}(x,\xi)đ_S\xi\,. \qquad (5.3.6)
\end{aligned}$$

Proof. See Section 5.6, the proof given there follows [37] (Let us point out here that the integral $\int_{T_x^*U} a_N(x,\xi)đ\xi$ converges because $a_N \in S^{\mathrm{Re}(\mu)-N-1+\varepsilon}$ for any $\varepsilon > 0$ and hence from the definition of the symbol classes we see that a_N is integrable over $T_x^*U \cong \mathbb{R}^n$.) □

An important property of the finite-part integral is that, in general, it is *not* invariant under a change of variable. The following Proposition is stated in [27] and gives precise information as to when the finite part integral can be defined globally on a manifold.

Proposition 5.3.4. *Let $a(x,\xi) \in \mathrm{LS}^{\nu,k}(U,V)$ be a log - polyhomogeneous symbol of order $\nu \in \mathbb{C}$ and log degree k. The following identity holds for any $A \in \mathrm{Gl}\left(T_x^*U\right)$:*

$$\begin{aligned}
\fint_{T_x^*U} a\big(x, A\xi\big)\ |A|đ\xi &= \fint_{T_x^*U} a\big(x,\xi\big)đ\xi \\
&\quad + \sum_{i=0}^{k} \frac{(-1)^{i+1}}{i+1} \int_{S_x^*U} a_{-n,i}(x,\xi) \log^{i+1}\left|A^{-1}\xi\right| đ\xi \qquad (5.3.7)
\end{aligned}$$

where $|A|$ denotes the determinant of A

Proof. See Section 5.7, the proof given there follows [27]. □

Corollary 5.3.5. *The finite - part integral in ξ of a log - polyhomogeneous symbol $a(x,\xi) \backsim \sum_{j=0}^{\infty}\sum_{i=0}^{k} a_{\mu-j,i}(x,\xi)\log^i[\xi]$ can be associated with a global density on the closed manifold M provided*

$$\sum_{i=0}^{k} \frac{(-1)^{i+1}}{i+1} \int_{S_x^*U} a_{-n,i}(x,\xi) \log^{i+1}\left|A^{-1}\xi\right| đ\xi = 0\,.$$

for any $A \in \mathrm{Gl}\left(T_x^*U\right)$

There are a number of examples in which this condition is satisfied (see also [37, Def. 1.1 and Prop. 1.10]):

Proposition 5.3.6. *Let* $a(x,\xi) \backsim \sum_{j\geq 0}\sum_{i=0}^{k_1} a_{\mu-j,i}(x,\xi) \in \mathrm{LS}^{\mu,k_1}(U,V)$ *and denote by* $A = \mathsf{Op}[a]$ *the corresponding operator. In each of the following cases*

$$\mathrm{TR}_x(A) := \fint_{T_x^*M} a(x,\xi)đ\xi\, dx \tag{5.3.8}$$

is a globally defined density on the factor M*:*

1. *the order* μ *is* not *an integer* $\geq n$ *(where* $n = \dim M$*)*

2. M *is odd - dimensional,* μ *is an integer and* a *is* even - even*, that is for each* $j \geq 0$*,*

   $$a_{\mu-j,i}(x,-\xi) = (-1)^{\mu-j} a_{\mu-j,i}(x,\xi)$$

   and this property also holds for all the derivatives of $a_{\mu-j,i}$*.*

3. M *is even - dimensional,* μ *is an integer and* a *is* even - odd*, that is for each* $j \geq 0$*,*

   $$a_{\mu-j,i}(x,-\xi) = (-1)^{\mu-j-1} a_{\mu-j,i}(x,\xi)$$

   and this property also holds for all the derivatives of $a_{\mu-j,i}$*.*

These results are all we need for now to study analogous definitions in our non-compact setting.

5.3.2 Finite-part integrals for simple warped products

First note that Lemma 5.3.1 carries over to address the divergence of $\int_{T_{(r,x)}\mathcal{O}} a(r,x,\xi)đ\xi$, let us state this here for later reference:

Lemma 5.3.7. *For* $a(r,x,\xi) \in \mathrm{L_2S}^{\nu,\mu,k_1,k_2}(\mathcal{O}, V)$ *and any* $N > \mathrm{Re}(\mu) + n + 1$ *one has*

$$\begin{aligned}
\fint_{T^*_{(r,x)}\mathcal{O}} a(r,x,\xi) đ\xi = \sum_{j=0}^{N-1} \int_{B^*_{(r,x)}(0,1)} a_{\mu-j}(r,x,\xi)\, đ\xi + \int_{T^*_{(r,x)}\mathcal{O}} a_{\mu-N}(r,x,\xi) đ\xi \\
+ \sum_{\substack{j=0 \\ \mu-j\neq-(n+1)}}^{N} \sum_{i=0}^{k_1} \frac{(-1)^{i+1} i!}{(\mu-j+n+1)^{i+1}} \int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\xi)\, đ_S\xi\,.
\end{aligned} \tag{5.3.9}$$

We substitute this expression into

$$\int_{[0,\infty)\times M} \Big(\fint_{T^*_{(r,x)}\mathcal{O}} a(r,x,\xi)\, đ\xi \Big) dx dr \tag{5.3.10}$$

which gives

$$\begin{aligned}
= \int_{[0,\infty)\times M} \Big(\sum_{j=0}^{N-1} \int_{B^*_{(r,x)}(0,1)} a_{\mu-j}(r,x,\xi) đ\xi + \int_{T^*_{(r,x)}\mathcal{O}} a_{\mu-N}(r,x,\xi) đ\xi \\
+ \sum_{\substack{j=0 \\ \mu-j\neq-(n+1)}}^{N} \sum_{i=0}^{k_1} \frac{(-1)^{i+1} i!}{(\mu-j+n+1)^{i+1}} \int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\xi) đ_S\xi \Big) dx dr
\end{aligned}$$

(note that the additional dimension yields an additional unit in the denominator of the factors in the second line). Using the representation $a_{\mu-j}(r,x,\xi) = \sum_{i=0}^{k_1} a_{\mu-j,i}(r,x,\xi) \log^i[\xi]$ $(0 \leq j \leq N)$ for the terms on the first line and rearranging the expression turns the above into

$$= \sum_{j=0}^{N-1} \sum_{i=0}^{k_1} \int_{[0,\infty)\times M} \Big(\int_{B^*_{(r,x)}(0,1)} a_{\mu-j,i}(r,x,\xi) \log^i[\xi]\, đ\xi \Big)\, dx dr \tag{5.3.11}$$

$$+ \sum_{i=0}^{k_1} \int_{[0,\infty)\times M} \Big(\int_{T^*_{(r,x)}\mathcal{O}} a_{\mu-N,i}(r,x,\xi) \log^i[\xi]\, đ\xi \Big)\, dx dr \tag{5.3.12}$$

$$+ \sum_{\substack{j=0 \\ \mu-j\neq-(n+1)}}^{N} \sum_{i=0}^{k_1} \frac{(-1)^{i+1} i!}{(\mu-j+n+1)^{i+1}} \int_{[0,\infty)\times M} \Big(\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\xi) đ_S\xi \Big)\, dx dr\,. \tag{5.3.13}$$

Finite part for the summands of line (5.3.11)

As in the proof for Lemma 5.3.3 we expand $a_{\mu-j,i}(r,x,\xi)$ into log - homogeneous components using (5.2.8) and substitute into the integral:

$$\begin{aligned}
&\int_{[0,\infty)\times M}\Big(\int_{B^*_{(r,x)}(0,1)} a_{\mu-j,i}(r,x,\xi)\log^i[\xi]\,đ\xi\Big)\,dxdr = \\
&\sum_{s=0}^{S-1}\int_{[0,\infty)\times M}\int_{B^*_{(r,x)}(0,1)} a_{\nu-s,\mu-j,i}(r,x,\xi)\log^i[\xi]\,đ\xi\,dxdr \\
&+\int_{[0,\infty)\times M}\int_{B^*_{(r,x)}(0,1)} a_{\nu-S,\mu-j,i}(r,x,\xi))\log^i[\xi]\,đ\xi\,dxdr
\end{aligned} \tag{5.3.14}$$

We choose S so that $a_{\nu-S,\mu-j,i}(r,x,\xi)$ is integrable in r, that is we need $\nu-\mathrm{Re}(S)<-1$. In this case the last term is finite as $R\to\infty$. The remaining terms are understood via the second log - homogeneous expansion (5.2.9),

$$\begin{aligned}
&\int_{[0,R]\times M}\int_{B^*_{(r,x)}(0,1)} a_{\nu-s,\mu-j,i}(r,x,\xi)\log^i[\xi]\,đ\xi\,dxdr \\
&=\int_{[0,R]\times M}\int_{B^*_{(r,x)}(0,1)}\Big(\sum_{l=0}^{k_2} a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r)\Big)\log^i[\xi]\,đ\xi\,dxdr\,.
\end{aligned} \tag{5.3.15}$$

Here the homogeneity property (5.2.10) allows us to split up and rewrite the integral as

$$\begin{aligned}
&=\int_{[0,1]\times M}\int_{B^*_{(r,x)}(0,1)}\Big(\sum_{l=0}^{k_2} a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r)\Big)\log^i[\xi]\,đ\xi\,dxdr \\
&+\int_{(1,R]\times M}\int_{B^*_{(r,x)}(0,1)}\Big(\sum_{l=0}^{k_2} a_{\nu-s,l,\mu-j,i}(f(1),x,\xi)\log^l f(r)\Big)\log^i[\xi]\,đ\xi\, f^{\nu-s}(r)dxdr
\end{aligned} \tag{5.3.16}$$

of which the first line is obviously finite. On the other hand, the second line contains divergent terms depending on f and therefore requires the application of a finite - part integral. The important point to note here is that the finite part will be completely determined by properties of the function f, in particular it is independent of the symbol $a(r,x,\xi)$.

Thus in full generality we have the following formula for the finite part integral of the first line in (5.3.11):

$$\sum_{j=0}^{N-1}\sum_{i=0}^{k_1} \fint_{[0,\infty)\times M}\Big(\int_{B^*_{(r,x)}(0,1)} a_{\mu-j,i}(r,x,\xi)\log^i[\xi]\,đ\xi\Big)\,dxdr$$
$$= \sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\int_M dx\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\int_0^1 dr\, a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r)$$
$$+\sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\int_M dx\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\int_0^\infty a_{\nu-S,\mu-j,i}(r,x,\xi))\,dr$$

(where S is chosen so that $\nu-\mathrm{Re}(S)<-1$, this ensures that $a_{\nu-S,\mu-j,i}(r,x,\xi)$ is integrable in r; so these are standard integrals, and)

$$+\sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\fint_{(1,\infty)\times M}\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\sum_{l=0}^{k_2} a_{\nu-s,l,\mu-j,i}(f(1),x,\xi)\log^l f(r)\; f^{\nu-s}(r)dxdr \tag{5.3.17}$$

which is a finite part integral.

Example 5.3.8. Suppose $f(r)=r$ for r sufficiently large, (say $r\geq 1/2$). Then (c.f. equations (5.6.5) and (5.6.6))

$$\left(\int_{[1,R)} r^{\nu-s}\log^l r\,dr\right)_{\nu-s=-1} = \frac{1}{l+1}\log^{l+1}R \tag{5.3.18}$$

and for $\nu-s\neq-1$,

$$\int_{[1,R)} r^{\nu-s}\log^l r\,dr = \sum_{p=0}^{l}\frac{(-1)^p l!/(l-p)!\log^{l-p}R}{(\nu-s+1)^{p+1}}R^{\nu-s+1}+\frac{(-1)^{l+1}l!}{(\nu-s+1)^{l+1}}\,. \tag{5.3.19}$$

The equation (5.3.18) and all terms on the right hand side of (5.6.6), except for the last, diverge as $R\to\infty$. Thus the finite part of the integral (5.3.16) in the case where $f(r)=r$ for $r\geq 1/2$ is

$$\big(1-\delta_{(\nu-s,-1)}\big)\int_M dx\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\sum_{l=0}^{k_2}\frac{(-1)^{l+1}l!}{(\nu-s+1)^{l+1}}\,a_{\nu-s,l,\mu-j,i}(1,x,\xi) \tag{5.3.20}$$

where $\delta_{n,m}$ denotes the Kronecker delta that evaluates to 1 if $n = m$ and otherwise to zero. We therefore obtain the following finite part for (5.3.11):

$$\sum_{j=0}^{N-1}\sum_{i=0}^{k_1} ⨍_{[0,\infty)\times M}\Big(\int_{B^*_{(r,x)}(0,1)} a_{\mu-j,i}(r,x,\xi)\log^i[\xi]\,đ\xi\Big)\,dxdr$$
$$= \sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\int_M dx\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\int_0^1 dr\,a_{\nu-s,l,\mu-j,i}(r,x,\xi)\log^l f(r)$$
$$+\sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\int_M dx\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\int_0^\infty dr\,a_{\nu-S,\mu-j,i}(r,x,\xi)$$

(where S is chosen so that $\nu - \mathrm{Re}(S) < -1$, this ensures that $a_{\nu-S,\mu-j,i}(r,x,\xi)$ is integrable in r; so these are standard integrals)

$$+\sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\big(1-\delta_{(\nu-s,-1)}\big)\int_M dx\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\ \times$$
$$\sum_{l=0}^{k_2}\frac{(-1)^{l+1}l!}{(\nu-s+1)^{l+1}}\,a_{\nu-s,l,\mu-j,i}(1,x,\xi)\,. \qquad (5.3.21)$$

Finite part for the summands of line (5.3.12)

Here the situation with respect to the integral over $[0,\infty)\times M$ is identical to that above. Thus for a general warping function $f(r)$ the formula for the finite part integral is given by

$$\sum_{i=0}^{k_1} ⨍_{[0,\infty)\times M}\Big(\int_{T^*_{(r,x)}\mathcal{O}} a_{\mu-N,i}(r,x,\xi)\log^i[\xi]\,đ\xi\Big)\,dxdr$$
$$= \sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\int_M dx\int_{T^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\int_0^1 dr\,a_{\nu-s,l,\mu-N,i}(f(r),x,\xi)\log^l f(r)$$
$$+\sum_{i=0}^{k_1}\int_M dx\int_{T^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\int_0^\infty dr\,a_{\nu-S,\mu-N,i}(r,x,\xi)$$

(where S is chosen so that $\nu - \mathrm{Re}(S) < -1$, this ensures that $a_{\nu-S,\mu-N,i}(r,x,\xi)$ is integrable; so these are standard integrals, plus the finite part integral below)

$$+\sum_{i=0}^{k_1}\sum_{s=0}^{S-1} \fint_{(1,\infty)\times M}\int_{T^*_{(r,x)}\mathcal{O}} \log^i[\xi]\, đ\xi \sum_{l=0}^{k_2} a_{\nu-s,l,\mu-N,i}(f(1),x,\xi)\log^l f(r)\ f^{\nu-s}(r)\, dxdr\,. \tag{5.3.22}$$

Example 5.3.8 (continued). For the case where $f(r) = r$ for $r \geq 1/2$ the finite part integral in (5.3.24) is equal to

$$\big(1-\delta_{(\nu-s,-1)}\big)\int_M dx \int_{T^*_{(r,x)}\mathcal{O}} \log^i[\xi]\, đ\xi \sum_{l=0}^{k_2} \frac{(-1)^{l+1} l!}{(\nu-s+1)^{l+1}} a_{\nu-s,l,\mu-N,i}(1,x,\xi)\,, \tag{5.3.23}$$

thus we obtain

$$\begin{aligned}
&\sum_{i=0}^{k_1}\fint_{[0,\infty)\times M}\Big(\int_{T^*_{(r,x)}\mathcal{O}} a_{\mu-N,i}(r,x,\xi)\log^i[\xi]\, đ\xi\Big)\, dxdr \\
&= \sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\int_M dx\int_{T^*_{(r,x)}\mathcal{O}}\log^i[\xi]\, đ\xi\int_0^1 dr\, a_{\nu-s,l,\mu-N,i}(f(r),x,\xi)\log^l r \\
&\qquad + \sum_{i=0}^{k_1}\int_M dx\int_{T^*_{(r,x)}\mathcal{O}}\log^i[\xi]\, đ\xi\int_0^\infty dr\, a_{\nu-S,\mu-N,i}(r,x,\xi)) \\
&+ \sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\big(1-\delta_{(\nu-s,-1)}\big)\int_M dx\int_{T^*_{(r,x)}\mathcal{O}}\log^i[\xi]\, đ\xi\sum_{l=0}^{k_2}\frac{(-1)^{l+1}l!}{(\nu-s+1)^{l+1}}\, a_{\nu-s,l,\mu-N,i}(1,x,\xi)\,.
\end{aligned} \tag{5.3.24}$$

Finally,

The summands of line (5.3.13)

Again the situation is similar. For a general warping function $f(r)$ the formula for the finite part integral of the third line is given by

$$\begin{aligned}
&\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\ \sum_{i=0}^{k_1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\fint_{[0,\infty)\times M}\Big(\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\xi)đ_S\xi\Big)\, dxdr \\
&\quad = \sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\int_M dx\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\, đ\xi
\end{aligned}$$

$$\times \int_0^1 dr\, a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r)$$

$$+\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\int_M dx\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\int_0^\infty dr\, a_{\nu-S,\mu-j,i}(r,x,\xi)$$

(the above are again standard integrals, and below we have the finite part contribution)

$$+\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}⨍_{(1,\infty)\times M}\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi$$

$$\times\sum_{l=0}^{k_2} a_{\nu-s,l,\mu-j,i}(f(1),x,\xi)\log^l f(r)\; f^{\nu-s}(r)dxdr \tag{5.3.25}$$

Example 5.3.8 (continued). Let us again look at the case where $f(r)=r$ for $r\geq 1/2$. Then we can compute each finite part integral in (5.3.27), we get

$$\big(1-\delta_{(\nu-s,-1)}\big)\int_M dx\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\sum_{l=0}^{k_2}\frac{(-1)^{l+1}l!}{(\nu-s+1)^{l+1}}a_{\nu-s,l,\mu-j,i}(1,x,\xi)\,, \tag{5.3.26}$$

thus

$$\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}⨍_{[0,\infty)\times M}\Big(\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\xi)đ_S\xi\Big)\,dxdr$$

$$=\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\int_M dx\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi$$

$$\times\int_0^1 dr\, a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r)$$

$$+\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\int_M dx\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\int_0^\infty dr\, a_{\nu-S,\mu-j,i}(r,x,\xi)$$

$$+\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\sum_{s=0}^{S-1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\big(1-\delta_{(\nu-s,-1)}\big)\int_M dx\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi$$

$$\sum_{l=0}^{k_2}\frac{(-1)^{l+1}l!}{(\nu-s+1)^{l+1}}a_{\nu-s,l,\mu-j,i}(1,x,\xi) \tag{5.3.27}$$

5.4 The extended canonical trace

Let us turn to the algebra of operators associated with the symbols defined above, and consider the extension of the canonical trace.

5.4.1 Existence of a trace density

The first question to address is the existence of a trace density, in particular we need to check whether an analogue to Corollary 5.3.5 holds true, and to what extend the newly added dimension in each cotangent space changes the requirements for global well - definedness. It turns out that the obstruction is of the same form, yet the relevant term in the asymptotic expansion of a local symbol moves one index down. Thus, having an additional dimension in the cotangent space changes the location of the relevant data in the asymptotic expansion in a linear manner. To see this one should compare the following result with Corollary 5.3.5:

Proposition 5.4.1. *For a fixed $r \in [0, \infty)$ the finite - part integral in ξ of a symbol $a(r, x, \xi) \in \mathrm{L}_2\mathrm{S}^{\nu,\mu,k_1,k_2}(\mathcal{O}, V)$ can be associated with a global density on the closed manifold $(M, f^2(r)g)$ if, for any $A \in \mathrm{Gl}(T^*_{(r,x)}\mathcal{O})$ we have*

$$\sum_{i=0}^{k_1} \frac{(-1)^{i+1}}{i+1} \int_{S^*_{(r,x)}\mathcal{O}} a_{-n-1,i}(r, x, \xi) \log^{i+1} \left|A^{-1}\xi\right| đ\xi = 0\,. \tag{5.4.1}$$

Proof. We need to establish that the formula for the finite part integral given in equation (5.3.9) is invariant under a change of coordinates if (5.4.1) holds. First,

$$\int_{B^*_{(r,x)}(0,R)} a\big(r, x, A\xi\big)\ |A| đ\xi = \int_{A^{-1}B^*_{(r,x)}(0,R)} a\big(r, x, \xi\big) đ\xi \tag{5.4.2}$$

where $A^{-1}B^*_{(r,x)}\big(0, R\big) = \{\xi \in T^*_{(r,x)}\mathcal{O} \colon\ |A^{-1}\xi| \leq R\}$. Substitute into the right hand side the presentation

$$a(r, x, \xi) = \sum_{j=0}^{N-1} a_{\mu-j}(r, x, \xi) + a_{\mu-N}(r, x, \xi)$$

so that

$$\begin{aligned}
&\int_{A^{-1}B^*_{(r,x)}(0,R)} a(r,x,\xi)đ\xi \\
&= \sum_{j=0}^{N-1} \int_{A^{-1}B^*_{(r,x)}(0,R)} a_{\mu-j}(r,x,\xi)đ\xi + \int_{A^{-1}B^*_{(r,x)}(0,R)} a_{\mu-N}(r,x,\xi)đ\xi \qquad (5.4.3)
\end{aligned}$$

with N chosen large enough so that the last integral below is finite as $R \to \infty$. In the limit, this integral is independent of A and equals

$$\int_{T^*_{(r,x)}\mathcal{O}} a_{\mu-N}(r,x,\xi)đ\xi\,.$$

As for the remaining terms we break up the computation as follows

$$\begin{aligned}
&\int_{A^{-1}B^*_{(r,x)}(0,R)} a_{\mu-j}(r,x,\xi)đ\xi \\
&= \underbrace{\int_{B^*_{(r,x)}(0,1)} a_{\mu-j}(r,x,\xi)đ\xi}_{\text{finite}} + \int_{A^{-1}B^*_{(r,x)}(0,R)\setminus B^*_{(r,x)}(0,1)} a_{\mu-j}(r,x,\xi)đ\xi\,, \qquad (5.4.4)
\end{aligned}$$

this is valid for all R large enough so that $B^*_{(r,x)}(0,1) \subset A^{-1}B^*_{(r,x)}(0,R)$. Now for the second integral in (5.4.4) let us denote

$$\widetilde{A^{-1}B^*}_{(r,x)}(0,R) := A^{-1}B^*_{(r,x)}(0,R) \setminus B^*_{(r,x)}(0,1)$$

and use the polylogarithmic expansion of $a_{\mu-j}(r,x,\cdot)$ given in (5.2.6) to obtain

$$\begin{aligned}
&\int_{\widetilde{A^{-1}B^*}_{(r,x)}(0,R)} a_{\mu-j}(r,x,\xi)đ\xi = \sum_{i=0}^{k_1} \int_{\widetilde{A^{-1}B^*}_{(r,x)}(0,R)} a_{\mu-j,i}(r,x,\xi)\log^i|\xi|\,đ\xi \\
&= \sum_{i=0}^{k_1} \int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\eta) \int_1^{R/\left|A^{-1}\eta\right|} r^{\mu-j+n}\log^i r\,drđ\eta
\end{aligned}$$

and substituting (5.6.5) for each term in the sum we get , if $\nu - j + n = -1$,

$$\begin{aligned}
&\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\eta) \int_1^{R/\left|A^{-1}\eta\right|} r^{\mu-j+n}\log^i r\,drđ\eta \\
&= \frac{1}{i+1}\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\eta)\log^{i+1}\left(R/\left|A^{-1}\eta\right|\right)đ\eta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{i+1}\sum_{k=0}^{i}(-1)^k\binom{i+1}{k}\left(\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\eta)\log^k\left|A^{-1}\eta\right|\,đ\eta\right)\cdot\log^{i+1-k}R\\
&\quad+\frac{(-1)^{i+1}}{i+1}\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\eta)\log^{i+1}\left|A^{-1}\eta\right|\,đ\eta\,. \qquad (5.4.5)
\end{aligned}
$$

Here only the last term remains finite as $R\to\infty$. On the other hand, If $\mu-j+n\neq -1$ we see from (5.6.6) that

$$
\begin{aligned}
&\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\eta)\int_1^{R/\left|A^{-1}\eta\right|} r^{\mu-j+n}\log^i r\,drđ\eta\\
&= \frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\eta)đ\eta \qquad (5.4.6)\\
&+\sum_{l=0}^{i}\frac{(-1)^l i!/(i-l)!}{(\mu-j+n+1)^{l+1}}\times\\
&\qquad\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\eta)\big(R/\left|A^{-1}\eta\right|\big)^{\mu-j+n+1}\log^{i-l}\big(R/\left|A^{-1}\eta\right|\big)đ\eta\,, \quad (5.4.7)
\end{aligned}
$$

again only the term in line (5.4.6) remains finite as $R\to\infty$, furthermore it is already present in the formula (5.3.6) for the finite part integral. In summary, the additional terms that are created by the change in variables arise by summing over i the expression in (5.4.5), as claimed. □

Next, the example cases that were found to satisfy the analogue to (5.4.1) in the context of closed manifolds carry over to our setting. Note that the question of global well - definedness is *not* concerned with the whole of $M\times[0,\infty)$, instead by "global" we mean in this context a fixed fibre $M\times\{r\}$.

Proposition 5.4.2. *Let $a(r,x,\xi)\backsim\sum_{j\geq 0}\sum_{i=0}^{k_1} a_{\mu-j,i}(r,x,\xi)\in \mathrm{L_2S}^{\nu,\mu,k_1,k_2}(\mathcal{O},V)$ and denote by $A=\mathsf{Op}[a]$ the corresponding operator. In each of the following cases*

$$
\mathrm{TR}_{(r,x)}(A)\,dx := \fint_{T^*_{(r,x)}\mathcal{M}} a(r,x,\xi)đ\xi\;dx \qquad (5.4.8)
$$

is a globally defined density on the factor M:

1. *the order μ is* not *an integer $\geq -n-1$ (where $n=\dim M$)*

2. *M is even - dimensional, μ is an integer and a is* even - even*, that is for each $j \geq 0$,*

$$a_{\mu-j,i}(r, x, -\xi) = (-1)^{\mu-j} a_{\mu-j,i}(r, x, \xi) \tag{5.4.9}$$

and this property also holds for all the derivatives of $a_{\mu-j,i}$.

3. *M is odd - dimensional, μ is an integer and a is* even - odd*, that is for each $j \geq 0$,*

$$a_{\mu-j,i}(r, x, -\xi) = (-1)^{\mu-j-1} a_{\mu-j,i}(r, x, \xi) \tag{5.4.10}$$

and this property also holds for all the derivatives of $a_{\mu-j,i}$.

Proof. In each of these cases the integrals in (5.4.1) vanish. Indeed if the order μ is not an integer, or less than $-n-1$ then the component a_{-n-1} in the asymptotic expansion (which appears in the integrand) is zero by definition. In the other cases the result follows from the symmetry of the integrand. $\square$

Remark 5.4.3. Even though (5.4.8) is similar to the usual trace density observed in [37] it is *not* the same since the integration here takes place over $T^*_{(r,x)}\widetilde{M}$ instead of $T^*_x M$, the former has an additional dimension that accounts for the radial direction.

Proposition 5.4.2 allows us to define a natural extension of the canonical trace as follows:

Definition 5.4.4. Let $a \in \mathrm{L}_2\mathrm{S}^{\nu,\mu,k_1,k_2}(\mathcal{O}, V)$ satisfy any of the properties listed in Proposition 5.4.2, let $A := \mathsf{Op}[a]$ denote the pseudodifferential operator defined by a. The canonical trace is defined by setting

$$\mathrm{TR}(A) := ⨍_{[0,\infty)\times M} \mathrm{TR}(A)_{(r,x)}\, dxdr \tag{5.4.11}$$

5.4.2 A formula in terms of integrals of strongly polyhomogeneous symbols over the fibre

Let us find a formula for the integral above in terms of the local symbol of the operator.

Theorem 5.4.5. *With the notation and assumptions of the previous definition the extended canonical trace admits the following formula:*

$$\begin{aligned}
&\mathrm{TR}(A) = \\
&\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\int_{[0,1)}\log^l f(r)\,dr\int_M\left(\fint_{T^*_{(r,x)}\mathcal{O}}\tilde{a}_{s,l}(r)(x,\lambda,\eta)\,đ\eta\,đ\lambda\right)dx \\
&+\sum_{l=0}^{k_2}\int_{[0,\infty)}\log^l f(r)dr\int_M\left(\fint_{T^*_{(r,x)}\mathcal{O}}\tilde{a}_{S,l}(r)(x,\lambda,\eta)\,đ\eta\,đ\lambda\right)dx \\
&+\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\fint_{[1,\infty)}\log^l f(r)\;f^{\nu-s}(r)dr\int_M\left(\fint_{T^*_{(r,x)}\mathcal{O}}\tilde{a}_{s,l}(1)(x,\lambda,\eta)\,đ\eta\,đ\lambda\right)dx
\end{aligned}$$

where pointwise in r the integrands $\tilde{a}_{s,l}$ for $0 \leq s \leq S$ and $0 \leq l \leq k_2$ admit an asymptotic expansion

$$\tilde{a}_{s,l}(r)(x,\lambda,\eta) \backsim q_{\mu-j}(r)(x,\lambda,\eta) \qquad (\lambda,\eta)\to\infty$$

in which each term is of the form

$$q_{\mu-j}(r)(x,\lambda,\eta) = \sum_{i=0}^{k_1} q_{\mu-j,i}(r)(x,\lambda,\eta)\log^i[(\lambda,\eta)]$$

where the coefficients $q_{\mu-j,i}(r)(x,\lambda,\eta)$ are strongly polyhomogeneous in the sense of Grubb and Seeley [16, Definition 1.1].

Proof. Most of the work has already been done in Section 5.3.2 and what is left is to put the pieces together. First, we expand the trace density (5.4.8) by using the formula in Lemma 5.3.7; which can then be rearranged in the sum shown in (5.3.11) - (5.3.13):

$$\begin{aligned}
\mathrm{TR}(A) &= \sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\fint_{[0,\infty)\times M}\Big(\int_{B^*_{(r,x)}(0,1)} a_{\mu-j,i}(r,x,\xi)\log^i[\xi]\,đ\xi\Big)\,dxdr \\
&+\sum_{i=0}^{k_1}\fint_{[0,\infty)\times M}\Big(\int_{T^*_{(r,x)}\mathcal{O}} a_{\mu-N,i}(r,x,\xi)\log^i[\xi]\,đ\xi\Big)\,dxdr \\
&+\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\fint_{[0,\infty)\times M}\Big(\int_{S^*_{(r,x)}\mathcal{O}} a_{\mu-j,i}(r,x,\xi)đ_S\xi\Big)\,dxdr \qquad (5.4.12)
\end{aligned}$$

Next we take the computation for the outer finite part integrals from (5.3.17), (5.3.24) and (5.3.27) (note that each of them refers to multiple lines). Rearranging the result gives the following expression for the canonical trace:

$$
\begin{aligned}
&\mathrm{TR}(A) = \\
&\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\int_{[0,1)\times M}\log^l f(r)\,drdx\Bigg(\sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\ a_{\nu-s,l,\mu-j,i}(f(r),x,\xi) \\
&+\sum_{i=0}^{k_1}\int_{T^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\ a_{\nu-s,l,\mu-N,i}(f(r),x,\xi) \\
&+\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\ a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\Bigg) \qquad (5.4.13)\\
&+\int_{[0,\infty)\times M}drdx\Bigg(\sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\ a_{\nu-S,\mu-j,i}(r,x,\xi) \\
&+\sum_{i=0}^{k_1}\int_{T^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\, a_{\nu-S,\mu-N,i}(r,x,\xi) \\
&+\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\, a_{\nu-S,\mu-j,i}(r,x,\xi)\Bigg) \qquad (5.4.14)\\
&+\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}⨍_{[1,\infty)\times M}\log^l f(r)\ f^{\nu-s}(r)dxdr\Bigg(\sum_{j=0}^{N-1}\sum_{i=0}^{k_1}\int_{B^*_{(r,x)}(0,1)}\log^i[\xi]\,đ\xi\ a_{\nu-s,l,\mu-j,i}(f(1),x,\xi) \\
&+\sum_{i=0}^{k_1}\int_{T^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\ a_{\nu-s,l,\mu-N,i}(f(1),x,\xi) \\
&+\sum_{\substack{j=0\\ \mu-j\neq-(n+1)}}^{N}\sum_{i=0}^{k_1}\frac{(-1)^{i+1}i!}{(\mu-j+n+1)^{i+1}}\int_{S^*_{(r,x)}\mathcal{O}}\log^i[\xi]\,đ\xi\ a_{\nu-s,l,\mu-j,i}(f(1),x,\xi)\Bigg) \qquad (5.4.15)
\end{aligned}
$$

(we note that the change of summation and integration is allowable here since the integrals ending in line (5.4.13) and (5.4.14) respectively are absolutely integrable whilst the last integral is summed over powers in $\log f(r)$ therefore the summands are linearly independent expressions and cannot cancel each other out.) The brackets in the three expressions ending respectively in line (5.4.13), (5.4.14) and (5.4.15) are finite part integrals of certain functions defined on $T^*_{(r,x)}\mathcal{O}$ that are polylogarithmic in ξ. In fact, if we look at the local asymptotic expansion

$$a(r,x,\xi) \backsim \sum_{j=0}^{\infty}\sum_{i=0}^{k_1}\sum_{s=0}^{\infty}\sum_{l=0}^{k_2} a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r)\log^i[\xi]$$

and keep s,l fixed we obtain, for a fixed r, the asymptotic expansion

$$a_{s,l}(r)(x,\xi)\log^l f(r) \backsim \sum_{j=0}^{\infty}\sum_{i=0}^{k_1} a_{\nu-s,l,\mu-j,i}(f(r),x,\xi)\log^l f(r)\log^i[\xi]$$

so we see that $a_{s,l}$ is a function that has a polylogarithmic expansion in ξ. However, it is not exactly in the class of log - polyhomogeneous symbols in the sense of Lesch. To see this, it is better to distinguish the frequency variable that corresponds to the r - variable by splitting ξ into $\xi = (\lambda,\eta)$ where $\eta \in \mathbb{R}^n$ parametrises the cotangent space that corresponds to the subspace $T_xU \hookrightarrow T_{(r,x)}\mathcal{O}$ (we recall that $\mathcal{O} = (a,b)\times U$ where $U \subset M$ is an open subset). Then the above expansion is of the form

$$\tilde{a}_{s,l}(r)(x,\lambda,\eta)\log^l f(r) \backsim \sum_{j=0}^{\infty}\sum_{i=0}^{k_1} a_{\nu-s,l,\mu-j,i}(f(r),x,\lambda,\eta)\log^l f(r)\log^i[(\lambda,\eta)]\,. \tag{5.4.16}$$

where each function $a_{\nu-s,l,\mu-j,i}(f(r),x,\lambda,\eta)$ is a strongly polyhomogeneous symbol as defined by G. Grubb and R. Seeley in [16]. There they establish (c.f. [16, Theorem 1.16]) that classical polyhomogeneous symbols in $n+1$ cotangent variables give strongly polyhomogeneous symbols in n cotangent variables - this is precisely the case that we have here. Therefore we obtain

$$\begin{aligned}
&\mathrm{TR}(A) = \\
&\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}\int_{[0,1)}\log^l f(r)\,dr\int_M\left(⨍_{T^*_{(r,x)}\mathcal{O}}\tilde{a}_{s,l}(r)(x,\lambda,\eta)\,đ\eta\, đ\lambda\right)dx\\
&+\sum_{l=0}^{k_2}\int_{[0,\infty)}\log^l f(r)dr\int_M\left(⨍_{T^*_{(r,x)}\mathcal{O}}\tilde{a}_{S,l}(r)(x,\lambda,\eta)\,đ\eta\, đ\lambda\right)dx\\
&+\sum_{s=0}^{S-1}\sum_{l=0}^{k_2}⨍_{[1,\infty)}\log^l f(r)\; f^{\nu-s}(r)dr\int_M\left(⨍_{T^*_{(r,x)}\mathcal{O}}\tilde{a}_{s,l}(1)(x,\lambda,\eta)\,đ\eta\, đ\lambda\right)dx
\end{aligned}$$

where the integrands have the desired properties. □

Remark 5.4.6. The formula derived above rests on the assumption that each of the integrands in (5.4.12) exhibits log - polyhomogeneous growth in the r - variable, a fact that is build into the definition of the symbols under consideration. On the other hand, it is not obvious that the sum on the right hand side (i.e. the sum of the integrals) exhibits log - polyhomogeneous growth in the r - variable, that is whether the right hand side defines a symbol corresponding to a family of pseudodifferential operators over M, parametrised in the variable r. Related to this question is the interchangeability of the integration in the x - variable and the r - variable, which should be investigated regardless of the fact that the order done above (first in x then in r) is perhaps more natural since we consider families of pseudodifferential operators over the factor M. In this context a Fubini - type theorem similar to [28, Theorem 1.3] needs to be established because the interchange involves a standard integral (in the x - variable) as well as a finite part integral (in the r variable).

5.5 Example: The Laplace Beltrami Operator

5.5.1 Preliminary formulas

Symbol of $\triangle$ on multiply warped products

In general one can write down the scalar Laplace - Beltrami operator $\triangle$ on an n-dimensional Riemannian manifold in local coordinates $(x^1, \ldots, x^n)$ as

$$\triangle = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x_j} \right) \tag{5.5.1}$$

where $\det g$ denotes the determinant of the matrix g representing the metric tensor locally, and $\left(g^{ij}\right)_{1 \leq i,j \leq n} = g^{-1}$ is the inverse, so that $g^{ij} g_{jk} = \delta^i_j$ with δ^i_j the Kronecker delta. As we have seen in Proposition 3.2.1 this formula simplifies in the context of a product manifold $I \times M$ (I is assumed one - dimensional) with metric

$dr^2 + g_r$ where g_r is a smooth one - parameter family of metrics on M to

$$\triangle = -\frac{\partial^2}{\partial r^2} - \frac{1}{2}\operatorname{tr}(g_r^{-1}\dot{g}_r)\frac{\partial}{\partial r} + \triangle_r \tag{5.5.2}$$

where $\triangle_r$ denotes the Laplace - Beltrami operator on $(\mathcal{M}, g_r)$ and $\dot{g}_r := \frac{\partial}{\partial r} g_r$.
For the moment we are interested only in the special case of a multiply warped product $M = I \times \mathcal{M}_1 \times \cdots \times \mathcal{M}_m$ with I denoting an open interval where and the metric is of the form

$$dr^2 + h_1^2(r)g_1 + \cdots + h_m^2(r)g_m\,. \tag{5.5.3}$$

Here each $h_i\colon I \to \mathbb{R}$ is a smooth positive function, and $(\mathcal{M}_1^{d_i}, g_i)$, $i = 1, \ldots, m$ are compact Riemannian manifolds (of dimension d_i respectively).

Corollary 5.5.1. *The scalar Laplace - Beltrami operator $\triangle$ on the Riemannian manifold defined by* (5.5.3) *is given by*

$$\triangle = -\frac{\partial^2}{\partial r^2} - \left(\sum_{i=1}^m d_i \frac{\dot{h}_i}{h_i}\right)\frac{\partial}{\partial r} + \sum_{i=1}^m \frac{1}{h_i^2}\triangle_i \tag{5.5.4}$$

with $\triangle_i$ the scalar Laplace - Beltrami operator on $(\mathcal{M}_i, g_i)$.

Proof. This is immediate from (5.5.2) applied to the case (5.5.3), using the fact that a scaling of the metric $g \mapsto h_i^2 g$ leads to an inverse scaling of the Laplace Beltrami operator so that $(\triangle_i)_r = h_i^{-2}(r)\,\triangle_i$. $\square$

Of course the above also gives us the symbol for the Laplace Beltrami operator. Let $n = d_1 + \cdots + d_m$ denote the dimension of the factor $\mathcal{M}_1 \times \cdots \times \mathcal{M}_m$ in M, choose local coordinates

$$\big((r,x),(\eta,\xi)\big) = (r, x_1^1, \ldots, x_{d_1}^1, \ldots, x_{d_m}^m, \eta, \xi_1^1, \ldots, \xi_{d_1}^1, \xi_1^2, \ldots, \xi_{d_m}^m)$$

for the cotangent bundle where at a point $(r,x) \in M$ the variable η corresponds to the direction tangent to the factor I, and the remaining variables are ordered according to the factors, that is $x_1^i, \ldots, x_{d_i}^i$ are coordinates for $\mathcal{M}_i$ for $1 \leq i \leq m$ and likewise for the ξ-variables. We shall sometimes use the shorthand $\xi_i := \big(\xi_1^i, \ldots, \xi_{d_i}^i\big)$,

$x_i := \big(x^i_1, \ldots, x^i_{d_i}\big)$. Substituting η for $-i\partial_r$ and ξ^k_l for $-i\partial_{x^k_l}$ in equation (5.5.4) gives

Corollary 5.5.2. *The symbol $\sigma_\triangle = \sigma_\triangle\big((r,x),(\eta,\xi)\big)$ of the scalar Laplace - Beltrami operator $\triangle$ on the Riemannian manifold defined in line* (5.5.3) *is given by*

$$\sigma_\triangle = \eta^2 - i\left(\sum_{k=1}^m d_k \frac{\dot{h}_k}{h_k}\right)\eta + \sum_{k=1}^m \frac{1}{h_k^2}\sigma_{\triangle_k} \tag{5.5.5}$$

where $\sigma_{\triangle_k} = \sigma_{\triangle_k}\big(x,\xi\big)$ denotes the symbol of the Laplace Beltrami operator on the factor $\mathcal{M}_i$, that is

$$\sigma_{\triangle_k} = \|\xi_k\|^2_{g_k} - i\sum_{l=1}^{d_i} b^k_l \xi^k_l \tag{5.5.6}$$

with $\|\xi_k\|^2_{g_k} = \sum_{s,l=1}^{d_k} g^{sl}_k \xi^k_s \xi^k_l$ *and* $b^k_l = \sum_{s=1}^{d_k}\left(\frac{\partial g^{sl}_k}{\partial x^k_s} + \frac{1}{2}\operatorname{tr}\left(g_k^{-1}\frac{\partial g_k}{\partial x^k_s}\right) g^{sl}_k\right)$.

In particular, the homogeneous terms in $\sigma_\triangle$ are

$$a_2 = \eta^2 + \sum_{k=1}^m\left(\frac{1}{h_k^2}\sum_{s,l=1}^{d_k} g^{sl}_k\xi^k_s\xi^k_l\right), \tag{5.5.7}$$

$$a_1 = -i\left(\sum_{k=1}^m d_k\frac{\dot{h}_k}{h_k}\right)\eta - i\sum_{k=1}^m\frac{1}{h_k^2}\sum_{l=1}^{d_k} b^k_l\xi^k_l \tag{5.5.8}$$

$$a_0 = 0\,. \tag{5.5.9}$$

In terms of the homogeneous components a_{k2}, a_{k1} and a_{k0} of the Laplacians $\triangle_k$ one has

$$a_2 = \eta^2 + \sum_{k=1}^m \frac{1}{h_k^2}a_{k2}\,, \tag{5.5.10}$$

$$a_1 = -i\left(\sum_{k=1}^m d_k\frac{\dot{h}_k}{h_k}\right)\eta - i\sum_{k=1}^m\frac{1}{h_k^2}a_{k1} \tag{5.5.11}$$

$$a_0 = 0\,. \tag{5.5.12}$$

Remark 5.5.3. Looking back at Definition 5.2.3 we notice that the terms above require the fixed function to be $f(r) = r$. This shows in particular that one needs to be able to set f independent from the warping function h.

Formal Complex Power of $\triangle$ on multiply warped products

Next we consider the formal aspects of the complex power

$$\triangle^{-s} = \int_\Gamma \lambda^{-s}\big(\triangle - \lambda\big)^{-1} đ\lambda \tag{5.5.13}$$

of the Laplacian.

Remark 5.5.4. The theory of complex powers of elliptic operators on noncompact manifolds was studied for example by B. Amman, R. Lauter, V. Nistor and A. Vasy in [1], by E. Schrohe in [39] and also by U. Battisti and S. Coriasco [2]

Here Γ is an infinite contour surrounding the spectrum of $\triangle$, and $đ\lambda = \frac{id\lambda}{2\pi}$. If we take a function f supported in a coordinate neighbourhood $U \subset M$ with coordinates (r,x) then this operator is understood to act by

$$\begin{aligned}\triangle^{-s} f(r,x) &= \int_\Gamma \lambda^{-s}\big(\triangle - \lambda\big)^{-1} f(r,x) đ\lambda \\ &= \int_\Gamma \int_U \int_{T^*_{(r,x)}U} e^{i[(r,x)-(v,y)]\cdot(\eta,\xi)} \lambda^{-s} \sigma\big((r,x),(\eta,\xi),\lambda\big) f(v,y)\, đ\eta đ\xi dv dy đ\lambda\end{aligned}$$

with $đ\eta đ\xi = (2\pi)^{-(n+1)} d\eta d\xi$. So, formally, the Schwartz kernel of $\triangle^{-s}$ is

$$k_s\big((r,x),(v,y)\big) = \int_{T^*_{(r,x)}U} e^{i[(r,x)-(v,y)]\cdot(\eta,\xi)} \left(\int_\Gamma \lambda^{-s}\sigma\big((r,x),(\eta,\xi),\lambda\big) đ\lambda\right) đ\eta đ\xi$$

with $\sigma\big((r,x),(\eta,\xi),\lambda\big)$ the symbol of the resolvent operator $\big(\triangle - \lambda\big)^{-1}$. At least formally, the local symbol $\sigma\big(s,(r,x),(\eta,\xi)\big)$ of the complex power operator $\triangle^{-s}$ is thus given by

$$\sigma\big(s,(r,x),(\eta,\xi)\big) = \int_\Gamma \lambda^{-s}\sigma\big((r,x),(\eta,\xi),\lambda\big) đ\lambda \tag{5.5.14}$$

where $\sigma\big((r,x),(\eta,\xi),\lambda\big)$ is the symbol of the resolvent $(\triangle - \lambda)^{-1}$. We suppose there is, analogously to the resolvent formalism on compact manifolds, a local asymptotic expansion

$$\sum_{j\geq 0} q_{-2-j} = \sum_{j\geq 0} q_{-2-j}\big((r,x),(\eta,\xi),\lambda\big) \quad \text{for } (\eta,\xi) \to \infty$$

for the symbol of the resolvent where the terms in the formal series are recursively constructed out of the homogeneous summands listed in (5.5.7) - (5.5.9), explicitly:

$$q_{-2} = \big(a_2 - \lambda\big)^{-1} \tag{5.5.15}$$

$$q_{-2-j} = -q_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!} \partial^{\mu}_{(\eta,\xi)} q_{-2-l} \cdot D^{\mu}_{(r,x)} a_{2-k} \qquad (j \geq 1)\,. \tag{5.5.16}$$

Before we move on to study the symbol of the complex power $\triangle^{-s}$ let us make the following observation.

Proposition 5.5.5. *For each $j \geq 0$ the function q_{-2-j} is a polynomial $P\big(q_{-2}\big)$ in q_{-2} whose coefficients are independent of λ. Moreover the coefficients are in turn polynomials in ξ with coefficients that are determined by a_2, a_1 and their (r,x)-derivatives:*

$$q_{-2-j} = \sum_{\textit{finite}} \alpha_k q_{-2}^k \qquad \textit{with} \quad \alpha_k = \sum_{\textit{finite}} c_\tau(\partial^{\beta}_{(r,x)} a_2, \partial^{\nu}_{(r,x)} a_1) \xi^{\tau}\,. \tag{5.5.17}$$

Proof. We show this by induction. The base case $j = 0$ is clear, taking the polynomial $P(x) = x$. Now suppose that

$$q_{-2-l} = \sum_{\text{finite}} \alpha_{p_l} q_{-2}^{p_l}\,. \tag{5.5.18}$$

for each $0 \leq l < j$ where each α_{p_l} is independent of λ. Substituting this into the terms on the right hand side of (5.5.16) we see that

$$\begin{aligned} q_{-2-j} &= -q_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!} \partial^{\mu}_{(\eta,\xi)} \left(\sum_{p_l} \alpha_{p_l} q_{-2}^{p_l} \right) D^{\mu}_{(r,x)} a_{2-k} \\ &= -q_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!} \left(\sum_{p_l} \sum_{\gamma \leq \mu} \binom{\mu}{\gamma} \partial^{\mu-\gamma}_{(\eta,\xi)} \alpha_{p_l} \cdot \partial^{\gamma}_{(\eta,\xi)} q_{-2}^{p_l} \right) D^{\mu}_{(r,x)} a_{2-k}\,. \end{aligned} \tag{5.5.19}$$

Now

$$\partial^{\gamma}_{(\eta,\xi)} r_{-2}^{p_l} = \frac{\partial^{|\gamma|}}{\partial \eta^{\gamma_0} \partial \xi_1^{\gamma_1} \cdots \partial \xi_n^{\gamma_n}} (a_2 - \lambda)^{-p_l}$$

$$= \frac{\partial^{|\gamma|}}{\partial \eta^{\gamma_0} \partial \xi_1^{\gamma_1} \cdots \partial \xi_{n-1}^{\gamma_{n-1}}} \left(\frac{\partial^{\gamma_n}}{\partial \xi_n^{\gamma_n}} (a_2 - \lambda)^{-p_l} \right) \tag{5.5.20}$$

where, for the term inside the brackets, one may use Faà di Bruno's formula [23, Theorem 1.3.2] to compute

$$\frac{\partial^{\gamma_n}}{\partial \xi_n^{\gamma_n}} (a_2 - \lambda)^{-p_l} = \sum_{k=1}^{\gamma_n} (-1)^k \frac{p_l!}{(p_l - k)!} q_{-2}^{p_l + k} \cdot \underbrace{B_{n,k}\big(\partial_{\xi_n} a_2, \partial^2_{\xi_n} a_2, \ldots, \partial^{\gamma_n - k + 1}_{\xi_n} a_2\big)}_{\text{Bell polynomial, independent of } \lambda} . \tag{5.5.21}$$

Here the Bell polynomials on the right are independent of λ (see Remark 5.5.6 below for additional information on the Bell polynomial). Thus in the first iteration of taking derivatives in (5.5.20) we produce a polynomial in q_{-2} with coefficients independent of λ. Assuming the computation has been performed for all derivatives with respect to ξ_j for $j > m$ the next differentiation yields again a polynomial in q_{-2} as one can see from the expansion

$$\begin{aligned} &\frac{\partial^{\gamma_m}}{\partial \xi_m^{\gamma_m}} \sum_s q_{-2}^s \alpha_s = \sum_s \sum_{t=0}^{\gamma_m} \binom{\gamma_m}{t} \partial^t_{\xi_m} q^s_{-2} \cdot \partial^{\gamma_m - t}_{\xi_m} \alpha_s \\ &= \sum_s \sum_{t=0}^{\gamma_m} \binom{\gamma_m}{t} \left(\sum_{k=1}^{t} (-1)^k \frac{s!}{(s-k)!} q_{-2}^{s+k} \cdot B_{m,k}\big(\partial_{\xi_m} a_2, \partial^2_{\xi_m} a_2, \ldots, \partial^{t-k+1}_{\xi_m} a_2\big) \right) \cdot \partial^{\gamma_m - t}_{\xi_m} \alpha_s . \end{aligned} \tag{5.5.22}$$

Thus the term $\partial^{\gamma}_{(\eta,\xi)} q_{-2}^{p_l}$ in (5.5.19) is a polynomial in q_{-2} with coefficients that are independent of λ. Substitution of this into (5.5.19) and rearranging, using the fact that the terms involved are all scalar valued and therefore commute, yields the desired property of q_{-2-j}. □

Remark 5.5.6 (Bell polynomials). The (partial) Bell polynomial $B_{n,k}$ is defined as

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}$$

where the sum ranges over all possible multi - indices $(j_1, j_2, \ldots, j_{n-k+1})$ such that $\sum_i j_i = k$ and $\sum_i i j_i = n$. These polynomials are related, for example, to the number of partitions of a set of size n as one can compute this as a sum over partial Bell polynomials:

$$\text{no. of partitions } = \sum_{k=1}^{n} B_{n,k}(1, 1, \ldots, 1)\,.$$

We are now ready to determine an asymptotic expansion for the local symbol of $\triangle^{-s}$. Substitute the resolvent symbol expansion $\sum_{j\geq 0} q_{-2-j}$ into (5.5.14) to find

Proposition 5.5.7. *The local symbol of $\triangle^{-s}$ has an asymptotic expansion*

$$\sigma\big(s,(r,x),(\eta,\xi)\big) \backsim \sum_{j\geq 0} b_{-2-j}(s,(r,x),(\eta,\xi)) \qquad \text{as } (\eta,\xi)\to\infty \tag{5.5.23}$$

where

$$b_{-2s}\big(s,(r,x),(\eta,\xi)\big) = \Big(\eta^2 + \sum_{i=1}^{m} \frac{1}{h_i^2} a_{i2}\Big)^{-s} \tag{5.5.24}$$

and, for $j \geq 1$,

$$\begin{aligned} &b_{-2s-j}\big(s,(r,x),(\eta,\xi)\big) \\ &= \sum_{p_j} \frac{(s+p_j-2)\cdots(s+1)s}{(p_j-1)!} \Big(\eta^2 + \sum_{i=1}^{m} \frac{1}{h_i^2} a_{i2}\Big)^{-s-p_j+1} \alpha_{p_j}\big((r,x),(\eta,\xi)\big) \end{aligned} \tag{5.5.25}$$

where the outer sum is finite and α_{p_j} is determined by the homogeneous components a_2, a_1 and their derivatives as described in Proposition 5.5.5 and the proof thereof, in particular the α_{p_j} are independent of s.

Proof. The leading term in the expansion for the symbol of the complex power is

$$b_{-2s}\big(s,(r,x),(\eta,\xi)\big) = \int_\Gamma \lambda^{-s} q_{-2}\, đ\lambda = \int_\Gamma \lambda^{-s}\big(a_2-\lambda\big)^{-1} đ\lambda \tag{5.5.26}$$

and (5.5.24) now follows directly from (5.5.10) and Cauchy's Integral Theorem applied pointwise to the integral

$$s \mapsto \int_\Gamma \lambda^{-s}\big(a_2\big((r,x),(\eta,\xi)\big)-\lambda\big)^{-1} đ\lambda\,.$$

For the remaining symbols we have

$$b_{-2s-j} = \int_\Gamma \lambda^{-s} q_{-2-j}\, đ\lambda = \sum_{p_j} \Big(\int_\Gamma \lambda^{-s}\big(a_2-\lambda\big)^{-p_j} đ\lambda\Big)\, \alpha_{p_j} \tag{5.5.27}$$

where the second equality follows from Proposition 5.5.5. But

$$\int_{\Gamma} \lambda^{-s}\big(a_2-\lambda\big)^{-p_j} đ\lambda = \frac{1}{(p_j-1)!}\int_{\Gamma} \lambda^{-s}\frac{\partial^{p_j-1}}{\partial\lambda^{p_j-1}}\big(a_2-\lambda\big)^{-1} đ\lambda \tag{5.5.28}$$

and using integration by parts,

$$= \frac{(-1)^{p_j-1}}{(p_j-1)!}\int_{\Gamma} \frac{\partial^{p_j-1}}{\partial\lambda^{p_j-1}}\lambda^{-s}\cdot\big(a_2-\lambda\big)^{-1} đ\lambda \tag{5.5.29}$$

(note the boundary is at infinity where the integrated term vanishes). This simplifies to

$$= \frac{(s+p_j-2)\cdots(s+1)s}{(p_j-1)!}\int_{\Gamma} \lambda^{-s-p_j+1}\cdot\big(a_2-\lambda\big)^{-1} đ\lambda \tag{5.5.30}$$

and from the Cauchy Integral Formula,

$$= \frac{(s+p_j-2)\cdots(s+1)s}{(p_j-1)!}\Big(a_2\big((r,x),(\eta,\xi)\big)\Big)^{-s-p_j+1} \tag{5.5.31}$$

hence the result follows by (5.5.10) and by substituting this into the right hand side of (5.5.27). □

5.5.2 The case of a simple warp

We are now ready to formally determine the terms in the asymptotic expansion of $\triangle^{-s}$ on a simple product manifold $I \times M$ with warped metric $g = dr^2 + h^2(r)g$. We first focus on the leading symbol, of course the remaining terms in the expansion have to be treated as well. In particular a more precise description of the functions α_{p_j} in Proposition 5.5.7 is required and the topic of current work. From (5.5.7) we see that

$$a_2 = \eta^2 + \frac{1}{h^2}\|\xi\|_g^2 .$$

Since there is only one factor $\mathcal{M}$ involved let us drop the subscript g in $\|\xi\|_g$. Substituting this into (5.5.24) gives

$$b_{-2s} = \Big(\eta^2 + \frac{1}{h^2}\|\xi\|^2\Big)^{-s} \tag{5.5.32}$$

which is homogeneous in (η, ξ) of degree $-2s$. As we are interested in the large scale behaviour we may assume that $\|(\eta, \xi)\| \gg 0$ (however we shall see that we need to distinguish between the cases $\eta = 0$ and $\eta \neq 0$), the behaviour in r as $r \to \infty$ depends of course on properties of the function h. Let us find an expansion for (5.5.32) in terms of r: if $\eta \neq 0$ then we can rewrite the right hand side as

$$b_{-2s} = (\eta^2)^{-s}\Big(1 + \frac{\mu}{h^2}\Big)^{-s}. \tag{5.5.33}$$

with $\mu := \|\xi\|^2/\eta^2$. Then, provided $f \to \infty$ as $r \to \infty$ we may assume $|h^{-2}\mu| < 1$ for r large enough and (ξ, η) fixed, this allows an application of the binomial series

$$(1+y)^c = \sum_{n=0}^{\infty} \binom{c}{n} y^n \tag{5.5.34}$$

which is valid for $|y| < 1$ and any complex number c. Here,

$$\binom{c}{n} = \frac{c(c-1)\cdots(c-n+1)}{n!} = \frac{\Gamma(c+1)}{n!\Gamma(c+1-n)}$$

is a generalised binomial coefficient. On the other hand, if $\eta = 0$ then (5.5.32) is equal to

$$\Big(\frac{1}{h^2}\|\xi\|^2\Big)^{-s} = \Big(\|\xi\|^{-2s}\Big)h^{2s} = \Big(\|\xi\|^{-2s}\Big)e^{2s\ln h}$$

and we can apply the exponential series. For easier reference let us summarise the considerations above in a proposition.

Proposition 5.5.8. *Let $h(r)$ be a smooth positive function on $I = [0, \infty)$ such that $h \to \infty$ as $r \to \infty$. Then the (formal) leading symbol of $\triangle^{-s}$ on the manifold $I \times \mathcal{M}$ with metric $dr^2 + h^2(r)g$ has the following series expansion as $r \to \infty$*

$$b_{-2s}\big(s, (h, x), (\eta, \xi)\big) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{\Gamma(1-s)}{n!\Gamma(1-s-n)} \frac{\|\xi/\eta\|^{2n}}{\eta^{2s}} h^{-2n} & \text{if } \eta \neq 0\,, \\ \\ \displaystyle\sum_{n=0}^{\infty} \Big(\frac{(2s)^n}{n!\|\xi\|^{2s}}\Big) \ln^n h & \text{if } \eta = 0\,. \end{cases} \tag{5.5.35}$$

Remark 5.5.9. We see that, in each case, the principal symbol of $\triangle^{-s}$ admits an asymptotic expansion as $r \to \infty$ of the form

$$b_{-2s}\big(s,(r,x),(\eta,\xi)\big) \backsim \sum_{n\geq 0} \tilde{b}_n\big(s,(h,x),(\eta,\xi)\big)$$

where

away from the hypersurface $\boldsymbol{\eta} = \mathbf{0}$: the terms $\tilde{b}_n\big(s,(h,x),(\eta,\xi)\big)$ decay polynomially in h (and therefore decay in r) as well as η while they increase in $\|\xi\|$. In particular the components are *separately* homogeneous in the cotangent directions ξ and η:

$$\tilde{b}_n\big(s,(h,x),(\eta,t\,\xi)\big) = t^{2n}\,\tilde{b}_n\big(s,(h,x),(\eta,\xi)\big) \tag{5.5.36}$$

whilst

$$\tilde{b}_n\big(s,(h,x),(t\eta,\xi)\big) = t^{-2(n+s)}\,\tilde{b}_n\big(s,(h,x),(\eta,\xi)\big)\,. \tag{5.5.37}$$

Furthermore we have homogeneity in h of the form

$$\tilde{b}_n\big(s,(th,x),(\eta,\xi)\big) = t^{-2n}\,\tilde{b}_n\big(s,(h,x),(\eta,\xi)\big) \tag{5.5.38}$$

on the hypersurface $\boldsymbol{\eta} = \mathbf{0}$: the components $\tilde{b}_n\big(s,(h,x),(0,\xi)\big)$ grow logarithmically in h and the behaviour in $\|\xi\|$ is determined by the complex parameter s, concretely we have homogeneity of degree $-2s$ in the cotangent directions ξ:

$$\tilde{b}_n\big(s,(h,x),(0,t\,\xi)\big) = t^{-2s}\,\tilde{b}_n\big(s,(h,x),(0,\xi)\big) \tag{5.5.39}$$

Note that a good choice of s regularises this series.

Let us list some examples of metrics that arise in applications.

Example 5.5.10 (Generalized metric cones). For the class of metrics

$$dr^2 + r^{2k} g$$

where $h(r) = r^k$ with $k \in \mathbb{Z} \setminus \{0\}$ we see that the leading symbol of the complex power $\triangle^{-s}$ has an r-expansion of the form

$$b_{-2s}\big(s,(h,x),(\eta,\xi)\big) \overset{(r \gneqq 0)}{=} \begin{cases} \displaystyle\sum_{n=0}^{\infty} \binom{-s}{n} \frac{\|\xi/\eta\|^{2n}}{\eta^{2s}} r^{-2kn} & \text{if } \eta \neq 0\,, \\ \\ \displaystyle\sum_{n=0}^{\infty} \Big(\frac{(2s)^n k}{n!\|\xi\|^{2s}}\Big) \ln^n r & \text{if } \eta = 0\,. \end{cases} \tag{5.5.40}$$

Here any choice of positive integer k falls into the class discussed in Proposition 5.5.8.

Example 5.5.11 (Funnel). Another interesting example is the metric

$$dr^2 + \cosh^2(r)d\theta^2$$

associated with a Funnel, that is a certain type of cylindrical end which arises for example in the spectral and scattering theory on infinite - area hyperbolic surfaces (see for example [19, **?**]). Since the expansion is applicable as r becomes large we can use the approximation $\cosh(r) \backsim \frac{1}{2}e^r$ and obtain

$$b_{-2s}\big(s,(h,x),(\eta,\xi)\big) \underset{(r \gg 0)}{\backsim} \begin{cases} \displaystyle\sum_{n=0}^{\infty} \binom{-s}{n} \frac{\|\xi/\eta\|^{2n}}{\eta^{2s}} 4^n e^{-2rn} & \text{if } \eta \neq 0\,, \\ \\ \displaystyle\sum_{n=0}^{\infty} \Big(\frac{(2s)^n}{n!\|\xi\|^{2s}}\Big) r^n & \text{if } \eta = 0\,. \end{cases} \tag{5.5.41}$$

(for the lower branch we also approximate $r - \ln 2 \approx r$).

Finally let us comment on an example that does *not* satisfy the conditions of Proposition 5.5.8.

Example 5.5.12 (Cigar Soliton). The Cigar soliton is a steady gradient Ricci soliton on $\mathbb{R}^2$. A smooth Riemannian manifold (M,g) is called a *Ricci soliton* if there exists a smooth vector field X such that the Ricci tensor Ric of the metric g satisfies the equation $\mathrm{Ric} + \frac{1}{2}L_X g = \rho g$ where $L_X g$ is the Lie derivative of g in the direction of X

and ρ is a constant. Such manifolds are generalisations of Einstein spaces and play an important role in the study of the Ricci flow. If $\rho = 0$ then the Ricci soliton is called *steady*, and if the vector field arises as the gradient of some smooth function f (called the potential function of the Ricci soliton) then one speaks of a *gradient Ricci soliton*. The Cigar soliton was the first example of a complete noncompact steady soliton on $\mathbb{R}^2$, discovered by R. Hamilton [20], its metric can be written in the form

$$dr^2 + \tanh^2(r) d\theta^2 .$$

Of course the function $h(r) = \tanh(r)$ does not tend to $+\infty$ as r becomes large, this was an important assumption in order to be able to apply the binomial series (5.5.34). However, in the limit we know that $\tanh(r) \approx 1$ and

$$b_{-2s}\big(s, (h, x), (\eta, \xi)\big) \underset{(r \gg 0)}{\backsim} \Big(\eta^2 + \|\xi\|^2\Big)^{-s} , \tag{5.5.42}$$

In other words, one "quickly" looses the warping effect.

5.6 Proof of Lemma 5.3.3

Expand $a(x, \xi)$ into log-homogeneous components as in (5.2.5) and substitute into the integral $\int_{B_x^*(0,R)} a(x, \xi) đ\xi$,

$$\int_{B_x^*(0,R)} a(x, \xi) đ\xi = \sum_{j=0}^{N} \int_{B_x^*(0,R)} a_{\mu-j}(x, \xi) đ\xi + \int_{B_x^*(0,R)} a_N(x, \xi) đ\xi . \tag{5.6.1}$$

From the fact that $a_N \in S^{\mathrm{Re}(\mu)-N-1+\varepsilon}$ for any $\varepsilon > 0$ we see that

$$\int_{B_x^*(0,R)} |a_N(x, \xi)| \, đ\xi \leq C_x \int_{B_x^*(0,R)} (1 + |\xi|)^{\mathrm{Re}(\mu)-N-1+\varepsilon} đ\xi$$

and the right hand side is finite as $R \to \infty$ provided we choose $N > \mathrm{Re}(\mu) + n$. It follows by comparison that the last term on the r.h.s of (5.6.1) is finite, we denote the limit by

$$\int_{T_x^*U} a_N(x, \xi) đ\xi := \lim_{R \to \infty} \int_{B_x^*(0,R)} a_N(x, \xi) đ\xi . \tag{5.6.2}$$

The remaining terms can be studied using the logarithmic expansion of the $a_{\mu-j}$. We have

$$\int_{B^*_x(0,R)} a_{\mu-j}(x,\xi)đ\xi = \underbrace{\int_{B^*_x(0,1)} a_{\mu-j}(x,\xi)đ\xi}_{\text{finite}} + \int_{B^*_x(0,R)\setminus B^*_x(0,1)} a_{\mu-j}(x,\xi)đ\xi \tag{5.6.3}$$

and from (5.2.6), denoting $\tilde{B}^*_x(1,R) := B^*_x(0,R) \setminus B^*_x(0,1)$,

$$\begin{aligned} \int_{\tilde{B}^*_x(1,R)} a_{\mu-j}(x,\xi)đ\xi &= \sum_{i=0}^{k} \int_{\tilde{B}^*_x(1,R)} a_{\mu-j,i}(x,\xi/|\xi|)\,|\xi|^{\mu-j}\log^i|\xi|\,đ\xi \\ &= \sum_{i=0}^{k} \int_{S^*_xU} a_{\mu-j,i}(x,\eta)đ_S\eta \cdot \left(\int_1^R r^{\mu-j+n-1}\log^i r\,dr\right) \end{aligned} \tag{5.6.4}$$

Now if $\mu - j = -n$ then

$$\int_1^R r^{\mu-j+n-1}\log^i r\,dr = \frac{1}{i+1}\int_1^R \frac{d}{dr}\log^{i+1} r\,dr = \frac{1}{i+1}\log^{i+1} R\,. \tag{5.6.5}$$

Otherwise, repeated integration by parts yields [1]

$$\begin{aligned} \int_1^R r^{\mu-j+n-1}\log^i r\,dr &= \frac{\log^i R}{\mu-j+n}R^{\mu-j+n} - \frac{i}{\mu-j+n}\int_1^R r^{\mu-j+n-1}\log^{i-1} r\,dr \\ &\;\;\vdots \\ &= \sum_{l=0}^{i} \frac{(-1)^l i!/(i-l)!\log^{i-l} R}{(\mu-j+n)^{l+1}}R^{\mu-j+n} + \frac{(-1)^{i+1}i!}{(\mu-j+n)^{i+1}}\,. \end{aligned} \tag{5.6.6}$$

Substituting this into (5.6.4) gives

$$\int_{\tilde{B}^*_x(1,R)} a_{\mu-j}(x,\xi)đ\xi = \sum_{i=0}^{k} \int_{S^*_xU} a_{\mu-j,i}(x,\eta)đ_S\eta \cdot \frac{1}{i+1}\log^i R \tag{5.6.7}$$

if $\mu - j = -n$ and this diverges as $R \to \infty$, otherwise

$$\int_{\tilde{B}^*_x(1,R)} a_{\mu-j}(x,\xi)đ\xi = \sum_{i=0}^{k} \int_{S^*_xU} a_{\mu-j,i}(x,\eta)đ_S\eta \cdot \left(\sum_{l=0}^{i} \frac{(-1)^l i!/(i-l)!R^{\mu-j+n}}{(\mu-j+n)^{l+1}}\log^{i-l} R\right)$$

[1] This result deviates from the corresponding statement in [37], however only in aspects that are irrelevant to the final formula.

$$+\sum_{i=0}^{k}\int_{S_x^*U} a_{\mu-j,i}(x,\eta)đ_S\eta\cdot\left(\frac{(-1)^{i+1}i!}{(\mu-j+n)^{i+1}}\right) \tag{5.6.8}$$

and we see that the l.h.s of the first line diverges as $R\to\infty$ whereas the second line remains finite. The asymptotic expansion for $\int_{B_x^*(0,R)} a_{\mu-j}(x,\xi)đ\xi$ and the formula for the constant term $K(x)$ in (5.3.4) are now obtained by substituting (5.6.7) respectively (5.6.4) for the summands in the first term on the r.h.s of (5.7.3).

5.7 Proof of Proposition 5.3.4

First,

$$\int_{B_x^*(0,R)} a\big(x,A\xi\big)\ |A|đ\xi=\int_{A^{-1}B_x^*(0,R)} a\big(x,\xi\big)đ\xi \tag{5.7.1}$$

where $A^{-1}B_x^*\big(0,R\big)=\{\xi\in T_x^*U\colon\ |A^{-1}\xi|\le R\}$. Substitute

$$a(x,\xi)=\sum_{j=0}^{N} a_{\mu-j}(x,\xi)+a_N(x,\xi)$$

on the right hand side,

$$\int_{A^{-1}B_x^*(0,R)} a(x,\xi)đ\xi=\sum_{j=0}^{N}\int_{A^{-1}B_x^*(0,R)} a_{\mu-j}(x,\xi)đ\xi+\int_{A^{-1}B_x^*(0,R)} a_N(x,\xi)đ\xi \tag{5.7.2}$$

with N chosen large enough so that the last integral below is finite as $R\to\infty$. In the limit, this integral is independent of A and equals (5.6.2). For the remaining terms we have

$$\int_{A^{-1}B_x^*(0,R)} a_{\mu-j}(x,\xi)đ\xi=\underbrace{\int_{B_x^*(0,1)} a_{\mu-j}(x,\xi)đ\xi}_{\text{finite}}+\int_{A^{-1}B_x^*(0,R)\setminus B_x^*(0,1)} a_{\mu-j}(x,\xi)đ\xi\,, \tag{5.7.3}$$

valid for all R large enough so that $B_x^*(0,1)\subset A^{-1}B_x^*(0,R)$. Denote $\widetilde{A^{-1}B^*}_x(0,R):=A^{-1}B_x^*(0,R)\setminus B_x^*(0,1)$. For the second term on the right hand side we use the poly-logarithmic expansion of $a_{\mu-j}(x,\cdot)$ given in (5.2.6) to obtain

$$\int_{\widetilde{A^{-1}B^*}_x(0,R)} a_{\mu-j}(x,\xi)đ\xi=\sum_{i=0}^{k}\int_{\widetilde{A^{-1}B^*}_x(0,R)} a_{\mu-j,i}(x,\xi)\log^i|\xi|\,đ\xi$$

$$= \sum_{i=0}^{k} \int_{S_x^*U} a_{\mu-j,i}(x,\eta) \int_1^{R/\left|A^{-1}\eta\right|} r^{\mu-j+n-1} \log^i r \, dr đ\eta$$

and substituting (5.6.5) for each term in the sum we get , if $\mu - j = -n$,

$$\begin{aligned}
&\int_{S_x^*U} a_{\mu-j,i}(x,\eta) \int_1^{R/\left|A^{-1}\eta\right|} r^{\mu-j+n-1} \log^i r \, dr đ\eta \\
&= \frac{1}{i+1} \int_{S_x^*U} a_{\mu-j,i}(x,\eta) \log^{i+1} \left(R/\left|A^{-1}\eta\right|\right) đ\eta \\
&= \frac{1}{i+1} \sum_{k=0}^{i} (-1)^k \binom{i+1}{k} \left(\int_{S_x^*U} a_{\mu-j,i}(x,\eta) \log^k \left|A^{-1}\eta\right| \, đ\eta \right) \cdot \log^{i+1-k} R \\
&\quad + \frac{(-1)^{i+1}}{i+1} \int_{S_x^*U} a_{\mu-j,i}(x,\eta) \log^{i+1} \left|A^{-1}\eta\right| \, đ\eta \,. \qquad (5.7.4)
\end{aligned}$$

Here only the last term remains finite as $R \to \infty$. If $\mu - j \neq -n$ we see from (5.6.5) that

$$\begin{aligned}
&\int_{S_x^*U} a_{\mu-j,i}(x,\eta) \int_1^{R/\left|A^{-1}\eta\right|} r^{\mu-j+n-1} \log^i r \, dr đ\eta \\
&= \frac{(-1)^{i+1} i!}{(\mu-j+n)^{i+1}} \int_{S_x^*U} a_{\mu-j,i}(x,\eta) đ\eta \\
&\quad + \sum_{l=0}^{i} \frac{(-1)^l i!/(i-l)!}{(\mu-j+n)^{l+1}} \int_{S_x^*U} a_{\mu-j,i}(x,\eta) \left(R/\left|A^{-1}\eta\right|\right)^{\mu-j+n} \log^{i-l} \left(R/\left|A^{-1}\eta\right|\right) đ\eta \,, \qquad (5.7.5)
\end{aligned}$$

again only the first term remains finite as $R \to \infty$, furthermore it is already a term present in the formula (5.3.6) for the finite part integral. In summary, the additional terms that arise due to the change in variable arise by summing over i the term in (5.7.4), as claimed.

Chapter 6

Concluding remarks

This thesis revolves around the study of Riemannian manifolds, whose metric is equipped with a high degree of symmetry, using tools from pseudodifferential operator theory and more generally asymptotic analysis. A central theme is the heat kernel on cohomogeneity one manifolds, two objects that individually appear in a number of areas in mathematics and theoretical physics. The aim is to illuminate their intersection and thereby seek out more explicit and refined results.

In Chapter 2 we analyse the sectional curvature asymptotics of a particular set of cohomogeneity one metrics found by Andrew Dancer and McKenzie Wang in their study of the Einstein equations via the Hamiltonian formalism. In Chapter 3 we study a non-standard asymptotic expansion for the heat kernel on cohomogeneity one manifolds. Even though it is known that the standard asymptotic expansion for the heat kernel carries geometric information, it does not explicitly describe the "warping effect" that is present in cohomogeneity one metrics, a property that is more easily accessible in the non-standard approach. The asymptotic expansion of the trace of the heat kernel is equivalent to the expansion of the trace of the resolvent operator as well as the spectral zeta function. In this regard Chapter 4 is concerned with the coefficients for the standard heat trace expansion in the context of compact Riemannian manifolds, we show that these can be calculated via the resolvent symbols in an elementary fashion. Finally Chapter 5 represents an extension of the canonical trace to the setting of non-compact cohomogeneity one manifolds and is work in progress.

Let us now outline concluding remarks for each Chapter.

In Chapter 2 we studied the new examples of non - compact cohomogeneity one Ricci - flat Einstein manifolds of dimension 10 and 11 mentioned above. For the construction of these metrics Dancer and Wang assume that the Lie group acts by isometries on the manifold such that the principal orbit has codimension one (this is the cohomogeneity one property), furthermore it is assumed that the Lie group is a product $(G_1/K_1) \times (G_2/K_2)$ of distinct isotropy irreducible spaces. The solutions found are associated to particular dimension pairs of these factors, namely $(2, 8)$, $(3, 6)$, and $(5, 5)$, and the metric is diagonal of the form $g = dt^2 + f_1^2(t)\bar{g}_1 + f_2^2(t)\bar{g}_2$ where $\bar{g}_i$ is a homogeneous background metric on the i^{th} component of the principal orbit. We establish more explicit forms for these metrics in order to study sectional curvature asymptotics for large values of t. Let us refer to the transverse part as the 'horizontal factor' and to the two components constituting the fibre as 'first vertical factor' and 'second vertical factor'. In all cases we observe that sectional curvature associated with a plane that is tangent to the horizontal and the second vertical factor vanishes, i.e. with respect to the horizontal component and the second vertical component the metric asymptotically approaches a product metric. Sectional curvature associated with a plane that is tangent to the principal orbit is given for large t by the sectional curvature associated with the product metric $g_t = f_1^2(t)\bar{g}_1 + f_2^2(t)\bar{g}_2$ in the fibres. In particular, this means that sectional curvature is non - positive (respectively non - negative) for large t whenever both factors have that property. On the other hand, if both factors have positive sectional curvature (say) then their product has tangent planes whose sectional curvature is zero, namely planes that arise as the span of a vector tangent to the first component and a vector tangent to the second component (we thank J. Lotay for a helpful discussion that helped to clarify the geometric interpretation of the result). Finally, sectional curvature associated with a plane that is tangent to the horizontal and the first vertical factor vanishes in the case of the dimension pair $(3, 6)$ (so that the metric approaches a product metric in this case with respect to the first vertical factor as well) whilst it is unbounded in the other dimension pairs, from below for the pair $(2, 8)$ and from above for the pair $(5, 5)$. The results presented in this Chapter provide further information about the particular Einstein metrics and are therefore not immediately suitable for further work. However, since there exists (up to now) no complete list of conditions for the existence of Einstein metrics on manifolds of dimension larger than four, well studied examples are useful as they contribute to a better understanding of potential obstructions to existence and uniqueness of Einstein metrics.

The work in Chapter 3 is concerned with the extension of a non - standard parametrix construction from simple warps to multiply warped products, thereby accommodating the metrics that are studied in Chapter 2. The main motivation for this project originated in the idea that the special structure of (multiply) warped products brings within reach an understanding of the heat trace coefficients, not only in terms of the geometry of the underlying space as a whole, but in terms of the underlying geometries of the factors as well as the warping functions. One reason that this is an interesting research project to pursue is that it serves as an extension of the refined knowledge present for plain product geometries. In this regard, having a parametrix for the heat kernel that better accounts for the warped geometry is essential and a first step in this direction. The goal of future work is to apply the parametrix and study the corresponding short time asymptotic expansion of the heat trace as this is where the heat coefficients arise. In particular the aim is to calculate the first coefficients of concrete examples such as the Hamilton cigar and the Bryant Soliton. These are warped metrics, we hope that we can also calculate examples for doubly warped metrics such as the Einstein metrics found by Dancer and Wang that we studied in Chapter 2. In general, an explicit computation of these coefficients is very complicated so we plan to invest a certain amount of time in investigating computer software that is suitable for supporting symbolic computation. We hope that, whilst computationally demanding, this will serve as an avenue into a better understanding of the heat kernel on warped products and are confident that this is attainable in particular on simple warped products.

In Chapter 4 we establish explicit formulae for the first resolvent symbols of certain Laplace operators and motivate the interest in these via concrete applications. To illustrate the elementary nature of the resolvent symbols the discussion is restricted to simple yet important examples, and computations have been restricted to those resolvent symbols that were essential for the particular application presented here (i.e. terms that are known to reflect the underlying geometry, respectively terms necessary to derive the index on a Riemann surface). The next step is to generalise the discussion, , using a formal symbolic calculus along the lines of [33], and establish similar expressions for a generic resolvent symbol associated to a suitable pseudodifferential operator and investigate the geometric meaning of these. In this regard an interesting class of pseudodifferential operators are classical pseudodifferential operators whose components in the local symbol expansion are homogeneous in the jets of the metric and the connection (the symbolic calculus presented in [33] was brought to our attention by S. Paycha who also suggested the class of operators mentioned here). One point of caution that should be kept in mind is that the elementary nature of the calculation does not solve the difficulty in interpreting the result, this will likely turn out challenging especially in higher dimensions. A further interesting extension of the work is to study the applicability of the resolvent symbols in the context of non - compact manifolds. This is particularly interesting in the context of the topics covered in Chapter 2 and 5. We hope that the restriction to warped products will be the right context to further explore the use of the resolvent symbols presented here.

Finally in Chapter 5 we propose an extension of the canonical trace to the setting of non compact simple warped products. A suitable class of symbols is defined which extends log - polyhomogeneous symbols and is assumed to also exhibit log - polyhomogeneous growth in the non - compact spacial direction (the 'radial' direction). After showing that the class is closed under symbol composition we investigate the existence of a family of trace densities associated with certain pseudodifferential operators defined over the fibres. These operators are associated with strongly homogeneous symbols. In particular cases, which are similar to those present in the context of closed manifolds, it is possible to define a canonical trace by applying a second cut off integral in order to deal with the divergence in the radial variable. The work described here is still ongoing, in particular the results are intermediary and subject to review and improvement. For instance, the formula derived in Theorem 5.4.5 rests on the assumption that each of the integrands in (5.4.12) exhibits log - polyhomogeneous growth in the r - variable, a fact that is build into the definition of the symbols under consideration. However, it is not obvious that the sum on the right hand side (i.e. the sum of the integrals) exhibits log - polyhomogeneous growth in the r - variable, that is whether the right hand side defines a symbol corresponding to a family of pseudodifferential operators over M, parametrised in the variable r. Related to this question is the interchangeability of the integration in the x - variable and the r - variable, which should be investigated regardless of the fact that the order done here (first in x then in r) is perhaps more natural since we consider families of pseudodifferential operators over the factor M. In this context a Fubini - type theorem similar to [28, Theorem 1.3] needs to be established because the interchange involves a standard integral (in the x - variable) as well as a finite part integral (in the r variable). Nevertheless these issues, the work reported in this chapter forms an integral part of the thesis because it describes the first steps of future work that aims at studying the methods of Chapter 4 in the context of warped products. The results so far greatly clarified many advantages that arise from the cohomogeneity one structure and we are confident that it provides a realistic context in which to consider extensions for traces. Going forward we also plan to investigate a slightly different approach to the extension, based on the observation that cohomogeneity one metrics often arise in quotient constructions. With respect to these one may wish to understand the properties of geometric operators (such as the Laplace Beltrami operator) and fundamental solutions to differential equations (such as the heat kernel). This idea is classical – for example the Poisson summation formula and its generalisation, the Selberg Trace formula, are typical instances where the quotient structure of the underlying space is exploited to simplify the computation of the heat trace. To this end we consider a calculus for pseudodifferential operators whose symbols are invariant with respect to a particular group action. Using this invariance as the defining property of a symbol class for pseudodifferential operators we consider the extension of regularised traces in this context. The findings from the latter approach will then be compared to future results arising from the approach initiated in Chapter 5.

Chapter 7

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